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Fourier method for higher dimensional inverse quasi-linear parabolic problem

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Abstract

In this work, higher-dimensional inverse quasi-linear parabolic problem was investigated. We demonstrated the solution by the Fourier approximation. The inverse problem was first examined by linearizing and then used implicit finite difference schema for the numerical solution.

KEYWORDS

inverse problem, nonlocal(periodic) boundary conditions, two dimensional heat problem

1 | INTRODUCTION

Inverse parabolic problems in two dimensional are appeared especially applications in chemical diffusion, heat transfer processes have been used a lot such as [1, 5], population, medical area, electrochemistry, engineering, chemical area, and plasma physics [3]. This kind of problems with nonlocal boundary conditions are not easy to study. There are many papers on finding analytical and numerical solutions of inverse coefficient problems with nonocal boundary conditions in one dimensional. In these papers, finite difference method, boundary element method, finite element method, and so on are examined to approximate numerical solutions.

Finding of the unknown function in a nonlinear parabolic equation is used frequently by many engineers and scientists [1, 4, 6–8].

In this study, Fourier method is used for the theoretical part and linearizing and finite difference methods are used for the numerical procedure.

The article consists of four parts, in Section 2, the existence for solution is showed by iteration. In Section 3, continuous dependence for the solution is examined. In Section 4, the numerical approximation is given.

Here $\Gamma := \{0 < x < \pi, 0 < t < T\}$, $\upsilon(x, y)$, $h(t)$, $g(x, y, t, w)$ are given functions. $\{q(t), w(x, y, t)\} \in C[0, T] \times (C^{2,1}(\Gamma) \cap C^{1,0}(\overline{\Gamma}))$ is the solution of the inverse problem.

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} - q(t)w + g(x, y, t, w), \quad (x, y, t) \in \Gamma, \tag{1}$$

$$w(x,y,0)=\upsilon(x,y),\quad x\in[0,\pi],y\in[0,\pi], \tag{2}$$

$$\begin{aligned} w(0,y,t)&=w(\pi,y,t),\quad y\in[0,\pi],t\in[0,T],\\ w(x,0,t)&=w(x,\pi,t),\quad x\in[0,\pi],t\in[0,T], \end{aligned} \tag{3}$$

$$\begin{aligned} w_x(0,y,t)&=w_x(\pi,y,t),\quad y\in[0,\pi],t\in[0,T],\\ w_y(x,0,t)&=w_y(x,\pi,t),\quad x\in[0,\pi],t\in[0,T], \end{aligned} \tag{4}$$

$$h(t)=\int_0^\pi\int_0^\pi xyw(x,y,t)dxdy,\quad t\in[0,T]. \tag{5}$$

2 | SOLUTION OF TWO DIMENSIONAL MODEL

$$\begin{aligned} w(x,y,t)=&\frac{w_0(t)}{4}\\ &+\sum_{m,n=1}^{\infty}(w_{cmn}(t)\cos(2mx)\cos(2ny)+w_{csmn}(t)\cos(2mx)\sin(2ny))\\ &+\sum_{m,n=1}^{\infty}(w_{scmn}(t)\sin(2mx)\cos(2ny)+w_{smn}(t)\sin(2mx)\sin(2ny)), \end{aligned}$$

where

$$\upsilon_0=\frac{4}{\pi^2}\int_0^\pi\int_0^\pi\upsilon(x,y)dxdy,\quad \upsilon_{cmn}=\frac{4}{\pi^2}\int_0^\pi\int_0^\pi\upsilon(x,y)\cos(2mx)\cos(2ny)dxdy,$$

$$\begin{aligned} \upsilon_{csmn}&=\frac{4}{\pi^2}\int_0^\pi\int_0^\pi\upsilon(x,y)\cos(2mx)\sin(2ny)dxdy,\\ \upsilon_{scmn}&=\frac{4}{\pi^2}\int_0^\pi\int_0^\pi\upsilon(x,y)\sin(2mx)\cos(2ny)dxdy, \end{aligned}$$

$$\upsilon_{smn}=\frac{4}{\pi^2}\int_0^\pi\int_0^\pi\upsilon(x,y)\sin(2mx)\sin(2ny)dxdy.$$

By using Fourier method, we have Fourier coefficients:

$$w_0(t)=w_0(0)e^{-\int_0^t q(s)ds}+\frac{4}{\pi^2}\int_0^t\int_0^\pi\int_0^\pi e^{-\int_\tau^t q(s)ds}g(\alpha,\beta,\tau,w)d\alpha d\beta d\tau, \tag{6}$$

$$\begin{aligned} w_{cmn}(t)=&w_{cmn}(0)e^{-\int_0^t[((2m)^2+(2n)^2)+q(s)]ds}+\frac{4}{\pi^2}\int_0^t\int_0^\pi\int_0^\pi e^{-\int_\tau^t[((2m)^2+(2n)^2)+q(s)]ds}\\ &\times g(\alpha,\beta,\tau,w)\cos(2m\alpha)\cos(2n\beta)d\alpha d\beta d\tau, \end{aligned}$$

$$\begin{aligned} w_{csmn}(t)=&w_{csmn}(0)e^{-\int_0^t[((2m)^2+(2n)^2)+q(s)]ds}+\frac{4}{\pi^2}\int_0^t\int_0^\pi\int_0^\pi e^{-\int_\tau^t[((2m)^2+(2n)^2)+q(s)]ds}\\ &\times g(\alpha,\beta,\tau,w)\cos(2m\alpha)\sin(2n\beta)d\alpha d\beta d\tau, \end{aligned}$$

$$w_{scmn}(t) = w_{scmn}(0)e^{-\int_0^t[((2m)^2+(2n)^2)+q(s)]ds} + \frac{4}{\pi^2}\int_0^t\int_0^\pi\int_0^\pi e^{-\int_\tau^t[((2m)^2+(2n)^2)+q(s)]ds}$$
$$\times g(\alpha,\beta,\tau,w)\sin(2m\alpha)\cos(2n\beta)d\alpha d\beta d\tau,$$

$$w_{smn}(t) = w_{smn}(0)e^{-\int_0^t[((2m)^2+(2n)^2)+q(s)]ds} + \frac{4}{\pi^2}\int_0^t\int_0^\pi\int_0^\pi e^{-\int_\tau^t[((2m)^2+(2n)^2)+q(s)]ds}$$
$$\times g(\alpha,\beta,\tau,w)\sin(2m\alpha)\sin(2n\beta)d\alpha d\beta d\tau.$$

Let us show the solution:

$$\begin{aligned} w(x,y,t) = &\frac{1}{4}\left(\upsilon_0 + \frac{4}{\pi^2}\int_0^t e^{-\int_\tau^t q(s)ds} f_0(\tau,w)d\tau\right) \\ &+ \sum_{m,n=1}^\infty \left(\upsilon_{cmn} + \frac{4}{\pi^2}\int_0^t e^{-\int_\tau^t[((2m)^2+(2n)^2)+q(s)]ds} g_{cmn}(\tau,w)d\tau\right)\cos(2mx)\cos(2ny) \\ &+ \sum_{m,n=1}^\infty \left(\upsilon_{csmn} + \frac{4}{\pi^2}\int_0^t e^{-\int_\tau^t[((2m)^2+(2n)^2)+q(s)]ds} g_{csmn}(\tau,w)d\tau\right)\cos(2mx)\sin(2ny) \\ &+ \sum_{m,n=1}^\infty \left(\upsilon_{scmn} + \frac{4}{\pi^2}\int_0^t e^{-\int_\tau^t[((2m)^2+(2n)^2)+q(s)]ds} g_{scmn}(\tau,w)d\tau\right)\sin(2mx)\cos(2ny) \\ &+ \sum_{m,n=1}^\infty \left(\upsilon_{smn} + \frac{4}{\pi^2}\int_0^t e^{-\int_\tau^t[((2m)^2+(2n)^2)+q(s)]ds} g_{smn}(\tau,w)d\tau\right)\sin(2mx)\sin(2ny), \end{aligned} \tag{7}$$

where $\upsilon_0 = w_0(0)e^{-\int_0^t q(s)ds}$, $\upsilon_{cmn} = w_{cmn}(0)e^{-\int_0^t[((2m)^2+(2n)^2)+q(s)]ds}$, $\upsilon_{csmn} = w_{csmn}(0)$ $e^{-\int_0^t[((2m)^2+(2n)^2)+q(s)]ds}$.

(K1) $h(t) \in C^1[0, T]$,

(K2) $\upsilon(x, y)\epsilon C^{1,1}([0, \pi]\times[0, \pi])$, $\upsilon(0, y) = \theta(\pi, y)$, $\upsilon_x(0, y) = \upsilon_x(\pi, y)$, $\upsilon(x, 0) = \upsilon(x, \pi)$, $\upsilon_y(x, 0) = \upsilon_y(x, \pi)$ and

$$\int_0^\pi\int_0^\pi xy\upsilon(x,y)dxdy = h(0),$$

(K3) $g(x, y, t, w)$ is provided following conditions:

(1)

$$\left|\frac{\partial g(x,y,t,w)}{\partial x} - \frac{\partial g(x,y,t,\widetilde{w})}{\partial x}\right| \le l(x,y,t)|w-\widetilde{w}|,$$

$$\left|\frac{\partial g(x,y,t,w)}{\partial y} - \frac{\partial g(x,y,t,\widetilde{w})}{\partial y}\right| \le l(x,y,t)|w-\widetilde{w}|,$$

$$\left|\frac{\partial^2 g(x,y,t,w)}{\partial x\partial y} - \frac{\partial^2 g(x,y,t,\widetilde{w})}{\partial x\partial y}\right| \le l(x,y,t)|w-\widetilde{w}|,$$

where $l(x, y, t) \in L_2(\Gamma)$, $b(x, y, t) \ge 0$,

(2) $g(x, y, t, w) \in C^{2,2,0}[0, \pi]$, $t \in [0, T]$,

(3) $g(x, y, t, w)|_{x=0} = g(x, y, t, w)|_{x=\pi}$, $g_x(x, y, t, w)|_{x=0} = g_x(x, y, t, w)|_{x=\pi}$, $g_y(x, y, t, w)|_{y=0} = g_y(x, y, t, w)|_{y=\pi}$, $g_{xy}(x, y, t, w)|_{x=0} = g_{xy}(x, y, t, w)|_{x=\pi}$, $g_{xy}(x, y, t, w)|_{y=0} = g_{xy}(x, y, t, w)|_{y=\pi}$,

(4) $g_0(t,w) + \frac{\pi^2}{4}\sum_{m,n=1}^\infty \frac{(2m)^2+(2n)^2}{mn} g_{smn}(t,w) \ne 0,\ 0 < M_0 \le \min_{(x,y,t)\in\overline{\Omega}}|g(x,y,t,w)|.$

Let us provided the following assumptions.

$\upsilon(0,\ y) = \upsilon(\pi, y),\ \upsilon(x,\ 0) = \upsilon(x, \pi),\ g(0,\ y,\ t,\ w) = g(\pi,\ y,\ t,\ w),\ g(x,\ 0,\ t,\ w) = g(x,\ \pi,\ t,\ w),$ $\upsilon(x, y) \in C^{3,3}([0, \pi] \times [0, \pi])$, and $g(x, y, t) \in C^{3,3}([0, \pi] \times [0, \pi]),\ \forall t \in [0, T]$ in $\overline{\Gamma}$.

From the condition (K1)–(K3) and (4), we have

$$\int_0^\pi \int_0^\pi xyw_t(x, t)dxdy = h'(t), \quad 0 \le t \le T. \tag{8}$$

$$q(t) = \frac{1}{h(t)} \left\{ \begin{array}{c} -h'(t) + \frac{\pi^2}{4} \int_0^\pi \int_0^\pi g(\alpha, \beta, \tau, w)d\alpha d\beta + \sum\limits_{m,n=1}^{\infty} \frac{1}{mn} \int_0^\pi \int_0^\pi g(\alpha, \beta, \tau, w)d\alpha d\beta \\ -\frac{\pi^2}{4} \sum\limits_{m,n=1}^{\infty} \frac{(2m)^2+(2n)^2}{mn} \theta_{smn} e^{-\int_\tau^t [((2m)^2+(2n)^2)+q(s)]ds} \\ -\sum\limits_{m,n=1}^{\infty} \frac{(2m)^2+(2n)^2}{mn} \int_0^t \int_0^\pi \int_0^\pi g(\alpha, \beta, \tau, w)e^{-\int_\tau^t [((2m)^2+(2n)^2)+q(s)]ds} d\alpha d\beta d\tau \end{array} \right\}. \tag{9}$$

Definition 1 Let $\{w(t)\} = \{w_0(t),\ w_{cmn}(t),\ w_{csmn}(t),\ w_{scmn}(t),\ w_{smn}(t),\ m,\ n = 1,\ \dots\}$ satisfy that the condition

$$\max_{0 \le t \le T} \frac{|w_0(t)|}{4} + \sum_{m,n=1}^{\infty} \left(\max_{0 \le t \le T} |w_{cmn}(t)| + \max_{0 \le t \le T} |w_{csmn}(t)| + \max_{0 \le t \le T} |w_{scmn}(t)| + \max_{0 \le t \le T} |w_{smn}(t)| \right) < \infty$$

by B.

$$\|w(t)\| = \max_{0 \le t \le T} \frac{|w_0(t)|}{4} + \sum_{m,n=1}^{\infty} \left(\max_{0 \le t \le T} |w_{cmn}(t)| + \max_{0 \le t \le T} |w_{csmn}(t)| + \max_{0 \le t \le T} |w_{scmn}(t)| + \max_{0 \le t \le T} |w_{smn}(t)| \right) \text{ be the norm}$$

where B is Banach space.

Theorem 1 *If the conditions* $(K1)$–$(K3)$ *be implemented. Then it has a unique solution.*

Proof. We can give the equations to (7):

$$\begin{aligned}
w_0^{(N+1)}(t) &= \upsilon_0 + \frac{4}{\pi^2} \int_0^t \int_0^\pi \int_0^\pi g(\alpha, \beta, \tau, w^{(N)})e^{-\int_\tau^t [((2m)^2+(2n)^2)+q^{(N)}(s)]ds} d\alpha d\beta d\tau, \\
w_{cmn}^{(N+1)}(t) &= \upsilon_{cmn} + \frac{4}{\pi^2} \int_0^t \int_0^\pi \int_0^\pi g(\alpha, \beta, \tau, w^{(N)})e^{-\int_\tau^t [((2m)^2+(2n)^2)+q^{(N)}(s)]ds} \cos(2m\alpha)\cos(2n\beta)d\alpha d\beta d\tau, \\
w_{csmn}^{(N+1)}(t) &= \upsilon_{csmn} + \frac{4}{\pi^2} \int_0^t \int_0^\pi \int_0^\pi g(\alpha, \beta, \tau, w^{(N)})e^{-\int_\tau^t [((2m)^2+(2n)^2)+q^{(N)}(s)]ds} \cos(2m\alpha)\sin(2n\beta)d\alpha d\beta d\tau, \\
w_{scmn}^{(N+1)}(t) &= \upsilon_{scmn} + \frac{4}{\pi^2} \int_0^t \int_0^\pi \int_0^\pi g(\alpha, \beta, \tau, w^{(N)})e^{-\int_\tau^t [((2m)^2+(2n)^2)+q^{(N)}(s)]ds} \sin(2m\alpha)\cos(2n\beta)d\alpha d\beta d\tau, \\
w_{smn}^{(N+1)}(t) &= \upsilon_{smn} + \frac{4}{\pi^2} \int_0^t \int_0^\pi \int_0^\pi g(\alpha, \beta, \tau, w^{(N)})e^{-\int_\tau^t [((2m)^2+(2n)^2)+q^{(N)}(s)]ds} \sin(2m\alpha)\sin(2n\beta)d\alpha d\beta d\tau.
\end{aligned} \tag{10}$$

According to the assumption of the theorem, we get $w^{(0)}(t) \in \mathbf{B}$, $t \in [0, T]$. Using Lipshitz, Hölder, and Bessel inequality, we obtain:

$$\|w^{(1)}(t)\|_{\mathbf{B}} = \max_{0 \le t \le T} \frac{|w_0^{(1)}(t)|}{4} + \sum_{m,n=1}^{\infty} \left(\max_{0 \le t \le T} |w_{cmn}^{(1)}(t)| + \max_{0 \le t \le T} |w_{csmn}^{(1)}(t)| + \max_{0 \le t \le T} |w_{scmn}^{(1)}(t)| + \max_{0 \le t \le T} |w_{smn}^{(1)}(t)| \right)$$

$$\leq \frac{|v_0|}{2} + \sum_{m,n=1}^{\infty} (|v_{cmn}| + |v_{csmn}| + |v_{scmn}| + |v_{smn}|)$$
$$+ \left(\frac{4\sqrt{T}}{\pi^2} + \frac{8\sqrt{T}}{3\pi} \right) \|b(x,y,t)\|_{L_2(\Gamma)} \|w^{(0)}(t)\|_B$$
$$+ \left(\frac{4\sqrt{T}}{\pi^2} + \frac{8\sqrt{T}}{3\pi} \right) M.$$

According to the theorem, we get $w^{(1)}(t) \in \mathbf{B}$. For N,

$$\begin{aligned} \|w^{(N+1)}(t)\|_{B_1} &= \max_{0\leq t\leq T} \frac{|w_0^{(N)}(t)|}{4} + \sum_{m,n=1}^{\infty} \Big(\max_{0\leq t\leq T} |w_{cmn}^{(N)}(t)| + \max_{0\leq t\leq T} |w_{csmn}^{(N)}(t)| \\ &\quad + \max_{0\leq t\leq T} |w_{scmn}^{(N)}(t)| + \max_{0\leq t\leq T} |w_{smn}^{(N)}(t)| \Big) \\ &\leq \frac{|v_0|}{2} + \sum_{m,n=1}^{\infty} (|v_{cmn}| + |v_{csmn}| + |v_{scmn}| + |v_{smn}|) \\ &\quad + \left(\frac{4\sqrt{T}}{\pi^2} + \frac{8\sqrt{T}}{3\pi} \right) \|l(x,y,t)\|_{L_2(\Gamma)} \|w^{(N)}(t)\|_B \\ &\quad + \left(\frac{4\sqrt{T}}{\pi^2} + \frac{8\sqrt{T}}{3\pi} \right) M. \end{aligned}$$

According to $w^{(N)}(t) \in \mathbf{B}$ and assumption of the theorem, we get $w^{(N+1)}(t) \in \mathbf{B}$,

$$\{w(t)\} = \{w_0(t), w_{cmn}(t), w_{csmn}(t), w_{scmn}(t), w_{smn}(t), \quad m,n = 1,2,\ldots\} \in \mathbf{B}.$$

We can give by equation to (9):

$$q^{(N+1)}(t) = \frac{1}{h(t)} \left\{ \begin{array}{c} -h'(t) + \frac{\pi^2}{4} \int_0^\pi \int_0^\pi g(\alpha,\beta,\tau,w^{(N)}) d\alpha d\beta + \sum_{m,n=1}^{\infty} \frac{1}{mn} \int_0^\pi \int_0^\pi g(\alpha,\beta,\tau,w^{(N)}) d\alpha d\beta \\ -\frac{\pi^2}{4} \sum_{m,n=1}^{\infty} \frac{(2m)^2+(2n)^2}{mn} \theta_{smn} e^{-\int_\tau^t [((2m)^2+(2n)^2)+q^{(N)}(s)] ds} \\ -\sum_{m,n=1}^{\infty} \frac{(2m)^2+(2n)^2}{mn} \int_0^t \int_0^\pi \int_0^\pi g(\alpha,\beta,\tau,w^{(N)}) e^{-\int_\tau^t [((2m)^2+(2n)^2)+q^{(N)}(s)] ds} d\alpha d\beta d\tau \end{array} \right\}. \tag{11}$$

For $N = 0$,

$$\|q^{(1)}(t)\|_{C[0,T]} = \frac{1}{h(t)} \left\{ \begin{array}{c} |h'(t)| + \left(\frac{\pi^2}{4} + \frac{\pi^5}{4\sqrt{6}} + \frac{2\pi}{\sqrt{6}} \right) M + \left(\frac{\pi^5}{4\sqrt{6}} + \frac{2\pi}{\sqrt{6}} \right) \|l(x,y,t)\|_{L_2(\Gamma)} \|w^{(0)}(t)\|_B \\ \frac{\pi^3}{2\sqrt{6}} \sum_{m,n=1}^{\infty} |(v_{smn})_x| + |(v_{smn})_y| \end{array} \right\}. \tag{12}$$

Hence $q^{(1)}(t) \in B$. For N, we obtain,

$$\|q^{(N+1)}(t)\|_{C[0,T]} = \frac{1}{h(t)} \left\{ \begin{array}{c} |h'(t)| + \left(\frac{\pi^2}{4} + \frac{\pi^5}{4\sqrt{6}} + \frac{2\pi}{\sqrt{6}} \right) M + \left(\frac{\pi^5}{4\sqrt{6}} + \frac{2\pi}{\sqrt{6}} \right) \|l(x,y,t)\|_{L_2(\Gamma)} \|w^{(N)}(t)\|_B \\ \frac{\pi^3}{2\sqrt{6}} \sum_{m,n=1}^{\infty} |(v_{smn})_x| + |(v_{smn})_y| \end{array} \right\}.$$

Since $w^{(N)}(t) \in B$, $q^{(N)}(t) \in B$.

For $N \to \infty$, $w^{(N+1)}(t)$, $q^{(N+1)}$ are converged.

$$
\begin{aligned}
w^{(1)}(t) - w^{(0)}(t) &= \frac{(w_0^{(1)}(t) - w_0^{(0)}(t))}{4} \\
&+ \sum_{k=1}^{\infty} [(w_{cmn}^{(1)}(t) - w_{cmn}^{(0)}(t)) + (w_{csmn}^{(1)}(t) - w_{csmn}^{(0)}(t)) \\
&+ (w_{scmn}^{(1)}(t) - w_{scmn}^{(0)}(t)) + (w_{smn}^{(1)}(t) - w_{smn}^{(0)}(t))] \\
&= \frac{1}{4}\left(\frac{4}{\pi^2}\int_0^t\int_0^\pi\int_0^\pi [g_{\alpha\beta}(\alpha,\beta,\tau,w^{(0)}) - g_{\alpha\beta}(\alpha,\beta,\tau,0)]e^{-\int_\tau^t[((2m)^2+(2n)^2)+q^{(1)}(s)]ds}d\alpha d\beta d\tau\right) \\
&+ \sum_{m,n=1}^{\infty}\frac{4}{\pi^2 mn}\int_0^t\int_0^\pi\int_0^\pi [g_{\alpha\beta}(\alpha,\beta,\tau,w^{(0)}) - g_{\alpha\beta}(\alpha,\beta,\tau,0)] \\
&\times e^{-\int_\tau^t[((2m)^2+(2n)^2)+q^{(1)}(s)]ds}\cos(2m\alpha)\cos(2n\beta)d\alpha d\beta d\tau \\
&+ \sum_{m,n=1}^{\infty}\frac{4}{\pi^2 mn}\int_0^t\int_0^\pi\int_0^\pi [g_{\alpha\beta}(\alpha,\beta,\tau,w^{(0)}) - g_{\alpha\beta}(\alpha,\beta,\tau,0)] \\
&\times e^{-\int_\tau^t[((2m)^2+(2n)^2)+q^{(1)}(s)]ds}\cos(2m\alpha)\sin(2n\beta)d\alpha d\beta d\tau \\
&+ \sum_{m,n=1}^{\infty}\frac{4}{\pi^2 mn}\int_0^t\int_0^\pi\int_0^\pi [g_{\alpha\beta}(\alpha,\beta,\tau,w^{(0)}) - g_{\alpha\beta}(\alpha,\beta,\tau,0)] \\
&\times e^{-\int_\tau^t[((2m)^2+(2n)^2)+q^{(1)}(s)]ds}\sin(2m\alpha)\cos(2n\beta)d\alpha d\beta d\tau \\
&+ \sum_{m,n=1}^{\infty}\frac{4}{\pi^2 mn}\int_0^t\int_0^\pi\int_0^\pi [g_{\alpha\beta}(\alpha,\beta,\tau,w^{(0)}) - g_{\alpha\beta}(\alpha,\beta,\tau,0)] \\
&\times e^{-\int_\tau^t[((2m)^2+(2n)^2)+q^{(1)}(s)]ds}\sin(2m\alpha)\sin(2n\beta)d\alpha d\beta d\tau \\
&+ \frac{1}{4}\left(\frac{4}{\pi^2}\int_0^t\int_0^\pi\int_0^\pi g_{\alpha\beta}(\alpha,\beta,\tau,0)e^{-\int_\tau^t[((2m)^2+(2n)^2)+q^{(1)}(s)]ds}d\xi d\eta d\tau\right) \\
&+ \sum_{m,n=1}^{\infty}\frac{4}{\pi^2 mn}\int_0^t\int_0^\pi\int_0^\pi f_{\alpha\beta}(\alpha,\beta,\tau,0)e^{-\int_\tau^t[((2m)^2+(2n)^2)+q^{(1)}(s)]ds}\cos(2m\alpha)\cos(2n\beta)d\alpha d\beta d\tau \\
&+ \sum_{m,n=1}^{\infty}\frac{4}{\pi^2 mn}\int_0^t\int_0^\pi\int_0^\pi f_{\alpha\beta}(\alpha,\beta,\tau,0)e^{-\int_\tau^t[((2m)^2+(2n)^2)+q^{(1)}(s)]ds}\cos(2m\alpha)\sin(2n\beta)d\alpha d\beta d\tau \\
&+ \sum_{m,n=1}^{\infty}\frac{4}{\pi^2 mn}\int_0^t\int_0^\pi\int_0^\pi f_{\alpha\beta}(\alpha,\beta,\tau,0)e^{-\int_\tau^t[((2m)^2+(2n)^2)+q^{(1)}(s)]ds}\sin(2m\alpha)\cos(2n\beta)d\alpha d\beta d\tau \\
&+ \sum_{m,n=1}^{\infty}\frac{4}{\pi^2 mn}\int_0^t\int_0^\pi\int_0^\pi f_{\alpha\beta}(\alpha,\beta,\tau,0)e^{-\int_\tau^t[((2m)^2+(2n)^2)+q^{(1)}(s)]ds}\sin(2m\alpha)\sin(2n\beta)d\alpha d\beta d\tau.
\end{aligned}
$$

Let some inequalities are implemented (Bessel, Hölder, Lipschitzs):

$$
\|w^{(1)}(t) - w^{(0)}(t)\|_B \le \left(\frac{4\sqrt{T}}{\pi^2} + \frac{8\sqrt{T}}{3\pi}\right)(\|l(x,y,t)\|_{L_2(\Gamma)}\|w^{(0)}(t)\|_B + M),
$$

where

$$
S = \left(\frac{4\sqrt{T}}{\pi^2} + \frac{8\sqrt{T}}{3\pi}\right)(\|l(x,y,t)\|_{L_2(\Gamma)}\|w^{(0)}(t)\|_B + M).
$$

Same estimations:

$$
\|q^{(1)}(t) - q^{(0)}(t)\|_{C[0,T]} \le \frac{K}{1-L}\|w^{(1)}(t) - w^{(0)}(t)\|_B\|l(x,y,t)\|_{L_2(\Gamma)},
$$

where

$$K = \frac{1}{h(t)}\left(\pi^4 + \frac{\pi^5}{4\sqrt{6}} + \frac{2\pi}{\sqrt{6}}\right),$$

$$L = \frac{1}{h(t)}\left(\frac{\pi^3|T|}{2\sqrt{6}}\left(\sum_{k=1}^{\infty}|(v_{smn})_x| + |(v_{smn})_y|\right) + \frac{2\pi|T|}{\sqrt{6}}M\right).$$

$$\|w^{(2)}(t) - w^{(1)}(t)\|_B \leq CS\|l(x,y,t)\|_{L_2(\Gamma)},$$

where

$$C = \left(\frac{4\sqrt{T}}{\pi^2} + \frac{8\sqrt{T}}{3\pi}\right)\left(1 + \frac{MK|T|}{1-L}\right).$$

For the step N:

$$\|q^{(N+1)}(t) - q^{(N)}(t)\|_{C[0,T]} \leq \frac{K}{1-L}\|w^{(N+1)}(t) - w^{(N)}(t)\|_B\|l(x,y,t)\|_{L_2(\Gamma)}$$

$$\|w^{(N+1)}(t) - w^{(N)}(t)\|_B \leq \frac{S\|l(x,y,t)\|^N_{L_2(\Gamma)}}{\sqrt{N!}}C^N, \tag{13}$$

where

The series is uniformly convergent. Therefore $w^{(N+1)}(t)$ and $q^{(N+1)}(t)$ are converged.

$$\lim_{N\to\infty} w^{(N+1)}(t) = w(t), \quad \lim_{N\to\infty} q^{(N+1)}(t) = q(t).$$

$$\|w(t) - w^{(N+1)}(t)\|^2_{\mathbf{B}} \leq 2\left[\frac{C\|l(x,y,t)\|_{L_2(\Gamma)}\|w(t) - w^{(N+1)}(t)\|^2_{\mathbf{B}}}{\sqrt{N!}}\right]^2$$
$$\times \exp 2\left(\frac{4\sqrt{T}}{\pi^2} + \frac{8\sqrt{T}}{3\pi}\right)^2 \|l(x,y,t))\|^2_{L_2(\Gamma)}. \tag{14}$$

We obtain $w^{(N+1)} \to w$, $q^{(N+1)} \to q$, $N \to \infty$.

Applying same operations:

$$|w(t) - v(t)| \leq C\left(\int_0^t\int_0^\pi\int_0^\pi l^2(\alpha,\beta,\tau)|w(\tau) - v(\tau)|^2 d\alpha d\beta d\tau\right)^{\frac{1}{2}}, \tag{15}$$

we get $w(t) = v(t)$ and $r(t) = q(t)$.

The proof is over. ■

3 | STABILITY OF THE SOLUTION

Theorem 2 *Under* $(K1)$–$(K3)$ *the solution* (q, w) *of the* (1)–(5) *depends continuously.*

Proof. Supposed $\Phi = \{v, h', g\}$ and $\overline{\Phi} = \{\overline{v}, \overline{h'}, g\}$ and positive constants $M, K_i, i = 1, 2$ such that

$$\|g\|_{C^{1,1,0}[\Gamma]} \leq M, \|\overline{g}\|_{C^{1,1,0}[\Gamma]} \leq M, \|v\|_{C^3[0,\pi]} \leq K_1, \|\overline{v}\|_{C^3[0,\pi]} \leq K_1,$$

$$\|h\|_{C^1[0,T]} \leq K_2, \|\overline{h}\|_{C^1[0,T]} \leq K_2.$$

Let we take $\|\Phi\| = (\|h\|_{C^1[0,T]} + \|v\|_{C^{1,1}[0,\pi]} + \|g\|_{C^{1,1,0}[\overline{\Gamma}]})$. Let (q, w) and $(\overline{q}, \overline{w})$ be solutions of (1)–(5):

$$w - \overline{w} = \frac{(v_0 - \overline{v_0})}{4} + \sum_{m,n=1}^{\infty} v_{cmn} e^{-\int_\tau^t [((2m)^2+(2n)^2)+q(s)]ds} \cos(2m\xi)\cos(2n\eta)$$

$$\sum_{m,n=1}^{\infty} \overline{v_{cmn}} e^{-\int_\tau^t [((2m)^2+(2n)^2)+\overline{q}(s)]ds} \cos(2m\xi)\cos(2n\eta)$$

$$+ \sum_{m,n=1}^{\infty} v_{csmn} e^{-\int_\tau^t [((2m)^2+(2n)^2)+q(s)]ds} \cos(2m\xi)\sin(2n\eta)$$

$$+ \sum_{m,n=1}^{\infty} \overline{v_{csmn}} e^{-\int_\tau^t [((2m)^2+(2n)^2)+\overline{q}(s)]ds} \cos(2m\xi)\sin(2n\eta)$$

$$+ \sum_{m,n=1}^{\infty} \overline{v_{scmn}} e^{-\int_\tau^t [((2m)^2+(2n)^2)+\overline{q}(s)]ds} \sin(2m\xi)\cos(2n\eta)$$

$$+ \sum_{m,n=1}^{\infty} v_{scmn} e^{-\int_\tau^t [((2m)^2+(2n)^2)+q(s)]ds} \sin(2m\xi)\cos(2n\eta)$$

$$+ \sum_{m,n=1}^{\infty} v_{smn} e^{-\int_\tau^t [((2m)^2+(2n)^2)+q(s)]ds} \sin(2m\xi)\sin(2n\eta)$$

$$+ \sum_{m,n=1}^{\infty} \overline{v_{smn}} e^{-\int_\tau^t [((2m)^2+(2n)^2)+\overline{q}(s)]ds} \sin(2m\xi)\sin(2n\eta)$$

$$+ \frac{1}{4}\left(\frac{4}{\pi^2}\int_0^t\int_0^\pi\int_0^\pi [g(\alpha,\beta,\tau,u) - g(\alpha,\beta,\tau,\overline{u})] e^{-\int_\tau^t [((2m)^2+(2n)^2)+q(s)]ds} d\alpha d\beta d\tau\right)$$

$$+ \frac{1}{4}\left(\frac{4}{\pi^2}\int_0^t\int_0^\pi\int_0^\pi g(\alpha,\beta,\tau,\overline{u}) \left(e^{-\int_\tau^t [((2m)^2+(2n)^2)+q(s)]ds} - e^{-\int_\tau^t [((2m)^2+(2n)^2)+\overline{q}(s)]ds}\right) d\alpha d\beta d\tau\right)$$

$$+ \sum_{m,n=1}^{\infty} \frac{4}{\pi^2}\int_0^t\int_0^\pi\int_0^\pi [g(\alpha,\beta,\tau,w) - g(\alpha,\beta,\tau,\overline{w})] e^{-[(2m)^2+(2n)^2](t-\tau)}$$

$$\times \cos(2m\alpha)\cos(2n\beta) d\alpha d\beta d\tau$$

$$+ \sum_{k=1}^{\infty} \frac{4}{\pi^2}\int_0^t\int_0^\pi\int_0^\pi g(\alpha,\beta,\tau,\overline{w}) \left(e^{-\int_\tau^t [((2m)^2+(2n)^2)+q(s)]ds} - e^{-\int_\tau^t [((2m)^2+(2n)^2)+\overline{q}(s)]ds}\right)$$

$$\times \cos(2m\alpha)\cos(2n\beta) d\alpha d\beta d\tau$$

$$+ \sum_{m,n=1}^{\infty} \frac{4}{\pi^2}\int_0^t\int_0^\pi\int_0^\pi [g(\alpha,\beta,\tau,w) - g(\alpha,\beta,\tau,\overline{w})] e^{-\int_\tau^t [((2m)^2+(2n)^2)+q(s)]ds}$$

$$\times \cos(2m\xi)\sin(2n\eta) d\alpha d\beta d\tau$$

$$+ \sum_{m,n=1}^{\infty} \frac{4}{\pi^2}\int_0^t\int_0^\pi\int_0^\pi g(\alpha,\beta,\tau,\overline{u}) \left(e^{-\int_\tau^t [((2m)^2+(2n)^2)+q(s)]ds} - e^{-\int_\tau^t [((2m)^2+(2n)^2)+\overline{q}(s)]ds}\right)$$

$$\times \cos(2m\xi)\sin(2n\eta) d\alpha d\beta d\tau$$

$$+ \sum_{m,n=1}^{\infty} \frac{4}{\pi^2}\int_0^t\int_0^\pi\int_0^\pi [g(\alpha,\beta,\tau,u) - g(\alpha,\beta,\tau,\overline{u})] e^{-\int_\tau^t [((2m)^2+(2n)^2)+q(s)]ds}$$

$$\times \sin(2m\alpha)\cos(2n\beta) d\alpha d\beta d\tau$$

$$+\sum_{m,n=1}^{\infty}\frac{4}{\pi^2}\int_0^t\int_0^\pi\int_0^\pi g(\alpha,\beta,\tau,\overline{u})\left(e^{-\int_\tau^t[((2m)^2+(2n)^2)+q(s)]ds}-e^{-\int_\tau^t[((2m)^2+(2n)^2)+\overline{q}(s)]ds}\right)$$
$$\times\sin(2m\alpha)\cos(2n\beta)d\alpha d\beta d\tau$$
$$+\sum_{m,n=1}^{\infty}\frac{4}{\pi^2}\int_0^t\int_0^\pi\int_0^\pi[g(\alpha,\beta,\tau,u)-g(\alpha,\beta,\tau,\overline{u})]e^{-\int_\tau^t[((2m)^2+(2n)^2)+q(s)]ds}$$
$$\times\sin(2m\alpha)\sin(2n\beta)d\alpha d\beta d\tau$$
$$+\sum_{m,n=1}^{\infty}\frac{4}{\pi^2}\int_0^t\int_0^\pi\int_0^\pi g(\alpha,\beta,\tau,\overline{u})\left(e^{-\int_\tau^t[((2m)^2+(2n)^2)+q(s)]ds}-e^{-\int_\tau^t[((2m)^2+(2n)^2)+\overline{q}(s)]ds}\right)$$
$$\times\sin(2m\alpha)\sin(2n\beta)d\alpha d\beta d\tau$$

we obtain:

$$|w-\overline{w}|^2\leq 2M_2^2\|\Phi-\overline{\Phi}\|^2\times\exp 2M_1^2\left(\int_0^t\int_0^\pi l^2(\xi,\eta,\tau)d\xi d\eta d\tau\right)$$

where $M_1=\max\left\{M_3,\left(\frac{4\sqrt{T}}{\pi^2}+\frac{8\sqrt{T}}{3\pi}\right)\right\}$, $M_2=\max\{M_5,M_6\}$.

For $\Phi\to\overline{\Phi}$ then $w\to\overline{w}$. Hence $q\to\overline{q}$. ■

4 | THE NUMERICAL EXAMINATION

Let us make linearization of the nonlinear terms:

$$w_t^{(n)}=w_{xx}^{(n)}+w_{yy}^{(n)}-q(t)w^{(n)}+g(x,y,t,w^{(n-1)}),\quad (x,y,t)\in\Gamma, \tag{16}$$

$$w^{(n)}(x,y,0)=\upsilon(x,y),\quad x\in[0,\pi],\quad y\in[0,\pi], \tag{17}$$

$$\begin{aligned}&w^{(n)}(0,y,t)=w^{(n)}(\pi,y,t),\quad y\in[0,\pi]\quad t\in[0,T],\\&w^{(n)}(x,0,t)=w^{(n)}(x,\pi,t),\quad x\in[0,\pi]\quad t\in[0,T],\end{aligned} \tag{18}$$

$$\begin{aligned}&w_x^{(n)}(0,y,t)=w_x^{(n)}(\pi,y,t),\quad y\in[0,\pi]\quad t\in[0,T],\\&w_y^{(n)}(x,0,t)=w_y^{(n)}(x,\pi,t),\quad x\in[0,\pi]\quad t\in[0,T].\end{aligned} \tag{19}$$

If we take $w^{(n)}(x,y,t)=u(x,y,t)$ and $g(x,y,t,w^{(n-1)})=\widetilde{g}(x,y,t)$. Then we have a linear problem:

$$u_t=u_{xx}+u_{yy}-q(t)u+\widetilde{g}(x,y,t),\quad (x,y,t)\in\Gamma, \tag{20}$$

$$u(x,y,0)=\upsilon(x,y),\quad x\in[0,\pi],\quad y\in[0,\pi], \tag{21}$$

$$\begin{aligned}&u(0,y,t)=u(\pi,y,t),\quad y\in[0,\pi],\quad t\in[0,T],\\&u(x,0,t)=u(x,\pi,t),\quad x\in[0,\pi],\quad t\in[0,T],\end{aligned} \tag{22}$$

$$\begin{aligned}&u_x(0,y,t)=u_x(\pi,y,t),\quad y\in[0,\pi],\quad t\in[0,T],\\&u_y(x,0,t)=u_y(x,\pi,t),\quad y\in[0,\pi],\quad t\in[0,T].\end{aligned} \tag{23}$$

$[0,\pi]^2\times[0,T]$ is divided to an $M^2\times N$ mesh with the step sizes $h=\pi/M$, $\tau=T/N$.

Let us take $u_{i,j}^k$, $\widetilde{g}_{i,j}^k$, and q^k that instead of $u(x_i, y_j, t_k)$, $\widetilde{g}(x_i, y_j, t_k)$, and $q(t_k)$, respectively.

Then we examine implicit finite-difference method for problem (20)–(23):

$$\frac{1}{\tau}(u_{i,j}^{k+1} - u_{i,j}^k) = \frac{1}{h^2}(u_{i-1,j}^{k+1} - 2u_{i,j}^{k+1} + u_{i+1,j}^{k+1}) + \frac{1}{h^2}(u_{i,j-1}^{k+1} - 2u_{i,j}^{k+1} + u_{i,j+1}^{k+1}) - q^k u_{i,j}^k + \widetilde{g}_{i,j}^{k+1}, \tag{24}$$

$$u_{i,j}^0 = \upsilon_i, \tag{25}$$

$$\begin{aligned} u_{0,j}^k &= u_{M+1,j}^k, u_{M+1,j}^k = \frac{u_{1,j}^k - u_{M,j}^k}{2} \\ u_{i,0}^k &= u_{i,M+1}^k, u_{i,M+1}^k = \frac{u_{i,1}^k - u_{i,M}^k}{2} \end{aligned} \tag{26}$$

.

If we integrate Equation (1) with respect to x and y from 0 to π and use (3), (4), and (5), we obtain

$$q(t) = \frac{-h'(t) + \int_0^\pi y u_x(\pi, y, t)dy + \int_0^\pi x u_y(x, \pi, t)dx + \int_0^\pi \int_0^\pi xy\widetilde{g}(x, y, t)dxdy}{h(t)}. \tag{27}$$

The approximation function of (27) is

$$q^{k+1} = \frac{\left[-((h^{k+2} - h^k)/\tau) + \left(\int_0^\pi y u_x(\pi, y, t)dy\right)^k + \left(\int_0^\pi x u_y(x, \pi, t)dx\right)^k + \left(\int_0^\pi \int_0^\pi xy\widetilde{g}(x, y, t)dxdy\right)^k\right]}{h^k},$$

where $h^k = h(t_k)$, $k = 0, 1, \ldots, N$.

We are talking about the fact that the integrals are found numerically according to Simpson's integration rule applying the central difference scheme.

$q^{k(s)}$, $u_{i,j}^{k(s)}$ are the values of q^k, $u_{i,j}^k$ at the sth iteration step, respectively. Then, $q^{k+1(s+1)}$ is obtained as follows

$$q^{k+1(s+1)} = \frac{\left[-((h^{k+2} - h^k)/\tau) + \left(\int_0^\pi y v_x(\pi, y, t)dy\right)^{k(s)} + \left(\int_0^\pi x v_y(x, \pi, t)dx\right)^{k(s)} + \left(\int_0^\pi \int_0^\pi xy\widetilde{g}(x, y, t)dxdy\right)^{k(s)}\right]}{h^k},$$

$$\begin{aligned} \frac{1}{\tau}\left(u_{i,j}^{k+1(s+1)} - u_{i,j}^{k+1(s)}\right) &= \frac{1}{h^2}\left(u_{i-1,j}^{k+1(s+1)} - 2u_{i,j}^{k+1(s+1)} + u_{i+1,j}^{k+1(s+1)}\right) \\ &+ \frac{1}{h^2}\left(u_{i,j-1}^{k+1(s+1)} - 2u_{i,j}^{k+1(s+1)} + u_{i,j+1}^{k+1(s+1)}\right) - q^{k(s+1)}u_{i,j}^{k(s)} + \widetilde{g}_{i,j}^{k+1}, \end{aligned} \tag{28}$$

$$u_{i,j}^0 = \phi_i, \tag{29}$$

$$\begin{aligned} u_{0,j}^{k+1(s)} &= u_{M+1,j}^{k+1(s)}, \quad u_{M+1,j}^{k+1(s)} = \frac{v_{1,j}^{k+1(s)} - v_{M,j}^{k+1(s)}}{2}, \\ u_{i,0}^{k+1(s)} &= u_{i,M+1}^{k+1(s)}, \quad u_{i,M+1}^{k+1(s)} = \frac{u_{i,1}^{k+1(s)} - u_{i,M}^{k+1(s)}}{2}. \end{aligned} \tag{30}$$

$u_{i,j}^{k+1(s+1)}$ is found. If the value difference between the two iterations reaches the expected tolerance, iterations are stopped.

Example 1 This sample explores finding the accurate solution

$$\{q(t), w(x, y, t)\} = \{\exp(t+1), (\cos 2(x+y)\exp(t)\}.$$

TABLE 1 The errors for function $w(x, y, 1/2)$

Mesh point	x	y	Accurate w	Error
1	0	0	1.6405	0.0121
10	0.3579	0.3579	0.2566	0.0021
20	0.7556	0.7556	−1.6303	0.0214
30	1.1532	1.1532	−0.2566	0.0011
40	1.5509	1.5509	1.6203	0.0284
50	1.9486	1.9486	0.2566	0.0357
60	2.3463	2.3463	−1.6203	0.0284
70	2.7409	2.7409	−0.2566	0.0173
80	3.1416	3.1416	1.6203	0.0147

TABLE 2 The error for function $q(t)$

t	Accurate q	Error
0.0955	1.0997	0.0111
0.1960	1.2153	0.0122
0.2965	1.3431	0.0135
0.4020	1.4844	0.0149
0.4975	1.6405	0.0165
0.5980	1.8130	0.0182
0.6985	2.0037	0.0201
0.7990	2.2144	0.0223
1	2.7047	0.0272

for the given functions

$$v(x, y) = (\cos 2(x + y)), h(t) = \frac{\pi^2}{4} \exp(t),$$

$$g(x, y, t, u) = 5w \exp(t) + \exp(2t + 1) \cos 2(x + y).$$

The step sizes are $h = 0.0393$, $\tau = 0.005$.
$q(t)$ was $|q^{k+1(s+1)} - q^{k+1(s)}| \leq h/200$.

Comparisons between the accurate solution and the numerical solution are given in Tables 1 and 2 and Figures 1–3 when $T = 1$.

5 | CONCLUSION

The problem of inverse of the temperature distribution in the higher dimensional quasi-linear parabolic equation by nonlocal conditions is taken. This inverse problem has been examined from both theoretically and numerically. In the theoretical part, the existence of the problem and the stability of the problem were examined. In the numerical part, the finite difference method is used. Periodic boundary conditions are used in this article. The Fourier method and the finite difference method are crucial of inverse problems for two-dimensional quasi-linear parabolic equations.

In future studies, different conditions and different methods will be covered for higher dimensional problems.

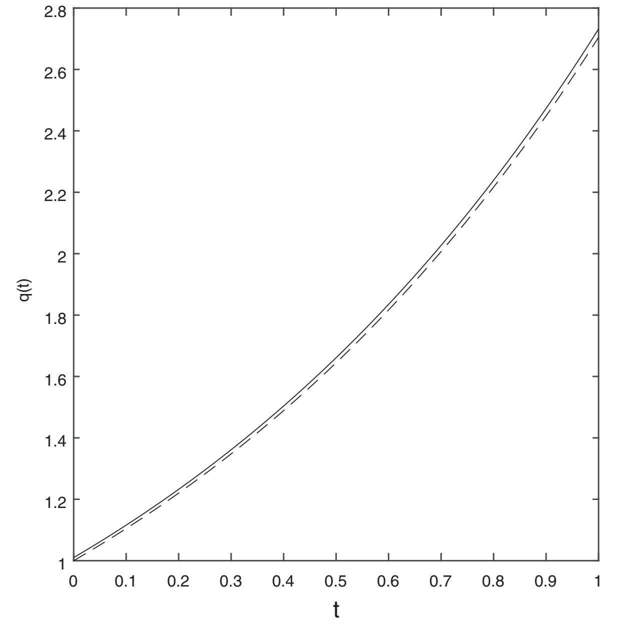

FIGURE 1 The accurate and numerical solutions of $q(t)$. The exact solution is shown with dashed line

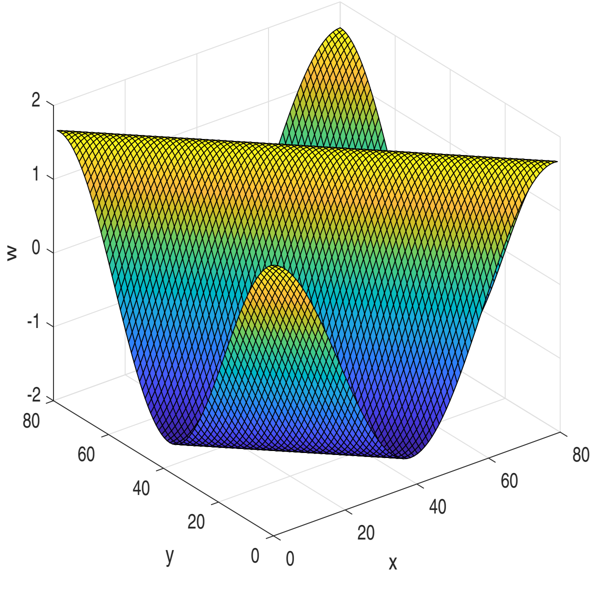

FIGURE 2 The numerical solution of $w(x, y, 1/2)$ [Color figure can be viewed at wileyonlinelibrary.com]

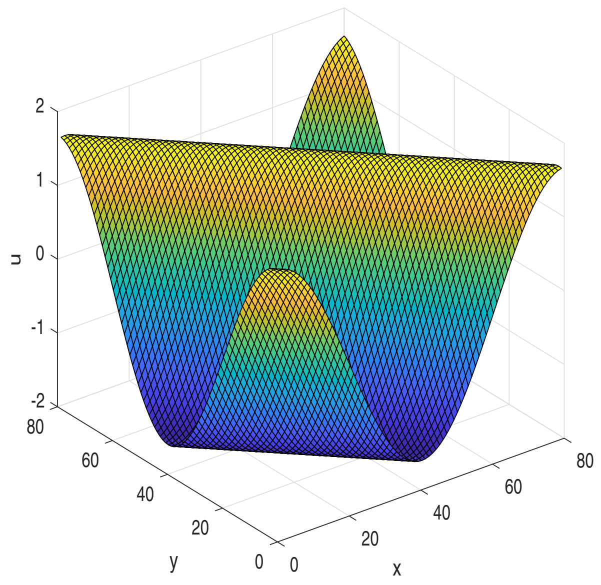

FIGURE 3 The accurate solution of $w(x, y, 1/2)$ [Color figure can be viewed at wileyonlinelibrary.com]

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