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FUNCTIONAL AND MATRIX APPROXIMATION OF NUMERICAL SOLUTION OF HAAR WAVELET

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Abstract

In this article, a uniform Haar wavelet approach is devised to numerically solve the differential equations. The uniform Haar wavelet coefficients are generated by employing collocation points. The generalized approach for function and matrix approximation using Haar wavelets is proposed. This study aims to decide which method is more useful by reflecting on the differences between the two methods. Also, the application of Haar wavelets to the solution of a first and second-order ODE is described in this research. To assess its applicability and efficiency, two test problems are used. The findings obtained are compared to those obtained using the function and matrix approximation methods. For numerically solving first and second-order ODEs, the Haar wavelet methodology gives a more reliable and exact method. By estimating error norms for various problems, the performance and accuracy of the method have been shown.

Key words: collocation points, Haar wavelet method, numerical methods
2020 Mathematics Subject Classification: 65T60, 65L10, 65R10

1. Introduction. In 1910, Haar [1] invented the Haar wavelet. The Haar wavelet is the most precise and efficient of all wavelet families. The Haar wavelet has been employed in several numerical methods for solving differential and integral equations [2, 3]. Using this technique, we obtain extensional Haar wavelet series, the highest order derivative of the differential equations. A comprehensive review of the HWM and its implementing are given in [4]. The formulation based on HWM was introduced, and the complexity difficulties of both the

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strong and weak formulation based on HWMs were examined in [5]. For lumped and distributed-parameter systems, CHEN and HSIAO [6] used the Haar wavelet method. In order to solve time-varying functional differential equations, HSIAO [7] employed a wavelet technique. Rationalized Haar functions were used by OHKITA and KOBAYASHI [8] to solve linear differential equations. CATTANI [9] proposed using Haar wavelet splines to solve differential equations numerically. LEPIK [10–12] employed Haar wavelets in his research for solving differential and integral equations. SUNMONU [13] demonstrated the use of maple to solve wavelet solutions for second-order differential equations.

In Section 2, short information about HWM is given. In Section 3, function approximation is used. Section 4 presents the solution obtained using the Haar wavelet collocation approach. In Section 5 there is a comparison of the numerical answers, and in Section 6 there are conclusions.

2. Haar wavelet family. The functions in the Haar wavelet family are constant functions achieving just three values 0, 1 and -1. For $t \in [0,1]$ they are defined as

$$\hbar_l(t) = \begin{cases} 1, & t \in [\xi_1, \xi_2) \\ -1, & t \in [\xi_2, \xi_3) \\ 0, & \text{elsewhere} \end{cases} \qquad \xi_1 = \frac{b}{c}, \quad \xi_2 = \frac{b+0.5}{c}, \quad \xi_3 = \frac{b+1}{c}.$$

The value of c is defined as $c = 2^k$, $k = 0, 1, 2, \ldots, K$, $b = 0, 1, 2, \ldots, c-1$, and b, K, c show the translation parameter, maximal level of resolution and level of wavelets, respectively. The value of subscript l is calculated by $l = c + b + 1$; through this formulation the minimum value of $l = 2$ for which $c = 1$, $b = 0$; maximal value of l is $l = 2C = 2^{K+1}$. The scaling function appears as follows when we get l to 1

$$\hbar_1(t) = \begin{cases} 1, & t \in [0,1) \\ 0, & \text{elsewhere} \end{cases}.$$

3. Function approximation. Haar wavelet series can be extended to each square integrable function $z(t)$ described on $[0,1)$ as follows:

$$z(t) = a_1\hbar_1(t) + a_2\hbar_2(t) + a_3\hbar_3(t) + \cdots = \sum_{l=1}^{\infty} a_l \hbar_l(t),$$

where the symbol a_l denotes the Haar wavelet coefficients and values $l = 1, 2, \ldots$ are given by

$$a_l = \int_0^1 z(t)\hbar_l(t)\,dt,$$

where $l = 2^k + b$, $k \geq 0$, $0 \leq b < 2^k$. Specially $a_1 = \int_0^1 z(t)\,dt$. $z(t)$ will be ended in finite terms if it can be approximated as a piecewise constant throughout each sub-interval, i.e.

$$z(t)=\sum_{l=1}^{C} a_l \hbar_l(t) = A_C^T H_C(t),$$

where the coefficients A_C^T and the Haar function vector $H_C(t)$ are defined for $c=2^k$ as $A_C^T=[a_1,\dots,a_C]$ and $H_C(t)=[\hbar_1(t),\dots,\hbar_C(t)]^T$.

4. Matrix application. Although Haar waves have some promising properties, the biggest disadvantage of these waves is their discontinuity at the break point, so the direct solution of differential equations using Haar waves is not possible. In order to get the better of this shortcoming, Chen and Hsiao [6] used the operation matrix approach primarily based on the Walsh function in the context of wavelet evaluation. The fundamental part of the method reduces the complexity of the problem by converting a set of differential equations into a set of algebraic equations.

Therefore, the integration of the vector

$$H_l(t)=[\hbar_1(t),\hbar_2(t),\dots,\hbar_l(t)]^T$$

can be obtained by Chen and Hsiao [6]

$$\int_0^t H_l(\rho)\,d\rho \cong QH_l(t)\,dt,$$

where Q is called Haar wavelet operational matrix of integration of order l. Yi et al. [14] suggested a new method that can use block pulse functions in a unified framework to derive the integral and differential operation matrices of all orthogonal functions

$$(\wp H)_{ls}=\int_0^{t_1} h_l(t)\,dt.$$

The following integrals are required in order to find the solution of a higher order differential equation

$$\wp_{l,1}(t)=\int_0^t \hbar_l(y)\,dy, \tag{4.1}$$

$$\wp_{l,\tau}(t)=\underbrace{\int_0^t\int_0^t\cdots\int_0^t}_{\tau\text{ times}} \hbar_l(y)\,dy, \tag{4.2}$$

where $\tau=2,3\dots$.

If equations (4.1) and (4.2) are used to calculate these integrals, then

$$\wp_{l,1}(t)=\begin{cases} t-\xi_1, & t\in[\xi_1,\xi_2)\\ \xi_3-t, & t\in[\xi_2,\xi_3)\\ 0, & \text{elsewhere}\end{cases},$$

$$\wp_{l,2}(t) = \begin{cases} 0, & t \in [0, \xi_1) \\ \frac{1}{2}(t-\xi_1)^2, & t \in [\xi_1, \xi_2) \\ \frac{1}{4m^2} - \frac{1}{2}(\xi_3 - t)^2, & t \in [\xi_2, \xi_3) \\ \frac{1}{4m^2}, & t \in [\xi_3, 1) \end{cases},$$

$$\wp_{l,\tau}(t) = \begin{cases} 0, & t < \xi_1 \\ \frac{(t-\xi_1)^\tau}{\tau!}, & t \in [\xi_1, \xi_2) \\ \frac{(t-\xi_1)^\tau}{\tau!} - 2\frac{(t-\xi_2)^\tau}{\tau!}, & t \in [\xi_2, \xi_3) \\ \frac{(t-\xi_1)^\tau}{\tau!} - 2\frac{(t-\xi_2)^\tau}{\tau!} + \frac{(t-\xi_3)^\tau}{\tau!}, & t \geq \xi_3 \end{cases},$$

$$\wp_{1,l}(t) = \frac{t^l}{l!}, \quad l = 1, 2, \dots .$$

The use of matrix formulation is convenient here. To this purpose the interval $t \in [0, 1]$ is separated into $2C$ parts of equal length $\Delta t = \frac{1}{2C}$, where $C = 2^K$. When we define the collocation points $t_s = \frac{(s-0.5)}{2C}$ $(s = 1, 2, \dots, 2C)$ and discretize the Haar function $\hbar_1(t)$, we get the coefficient matrix $H(l, s) = \hbar_l(t_s)$ which has dimension $2C \times 2C$. The integral matrices $\wp_\tau$ have elements $\wp_\tau(l, s) = \wp_{l,\tau}(s)$. The integral matrix was defined differently by Chen and Hsiao [6]. They introduced the row vector

$$\hbar_\mu(t) = [\hbar_1(t), \hbar_2(t), \dots, \hbar_\mu(t)], \tag{4.3}$$

where $\mu = 2C = 2^{K+1}$. Now we have

$$\int_0^t \hbar_{(\mu)}(\theta)\, d\theta \approx \wp_{(\mu\times\mu)} \hbar_{(\mu)}(t). \tag{4.4}$$

The operational matrix of integration is denoted as the square matrix $\wp_{(\mu\times\mu)}$. In system analysis, this matrix is a crucial method for assessing numerical solutions. The recursive formula shown below was proved by Chen and Hsiao

$$\wp_{(2\mu\times 2\mu)} = \frac{1}{4\mu} \begin{bmatrix} 4\mu\wp_{(\mu\times\mu)} & -H_{(\mu\times\mu)} \\ H^{-1}{}_{(\mu\times\mu)} & 0_{(\mu\times\mu)} \end{bmatrix} \tag{4.5}$$

whereas $\wp_{(1\times 1)} = 0.5$.

First order differential equations are solved using Equations (4.3)–(4.5). The order of the Haar matrices is increased by reducing higher-order equations to a set of first-order equations. The entries of the matrices H and $\wp$ can be evaluated. For instance when $C = 2$, it is obtained that

$$H_{4\times 4} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \quad \wp_{4\times 4} = \frac{1}{16} \begin{bmatrix} 8 & -4 & -2 & -2 \\ 4 & 0 & -2 & 2 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}.$$

5. Numerical application. In this section, by applying the previously mentioned HWM and HWCM, a first and second-order ODEs are solved. It is shown with error tables and graphs which of them provides a better approach to the exact solution.

Example 5.1. By taking the following equation

$$z'(t) + tz(t) = 4t^3, \quad z(0) = 1$$

with exact solution $z(t) = 4t^2 - 8 + 9e^{-\frac{t^2}{2}}$. For this equation, we compared the numerical approaches obtained by applying HWCM and HWM with the exact solution and discussed them in Table 1 which gives us error analysis of HWM and HWCM.

Table 1

Demonstration of absolute error for HWM/HWCM evaluation for $K = 3$

t_s	$z(t)$	$\lvert z(t_s) - z(t)\rvert$ with HWM	$\lvert z(t_s) - z(t)\rvert$ with HWCM
0.03125	0.999512791	$4.84589781051259 \times 10^{-4}$	$6.82202285968747 \times 10^{-3}$
0.09375	0.995692245	$4.48598607090456 \times 10^{-4}$	$1.41506920415625 \times 10^{-2}$
0.15625	0.988460801	$3.77698020401884 \times 10^{-4}$	$1.41494700215625 \times 10^{-2}$
0.21875	0.978629791	$2.74013177870458 \times 10^{-4}$	$1.40470083409375 \times 10^{-2}$
0.28125	0.967396516	$1.40634063335199 \times 10^{-4}$	$1.37446388071251 \times 10^{-2}$
0.34375	0.956321218	$1.84874672908286 \times 10^{-5}$	$1.31469631702500 \times 10^{-2}$
0.40625	0.947297331	$1.98687932640107 \times 10^{-4}$	$1.21916021196875 \times 10^{-2}$
0.46875	0.942515839	$3.94748238971676 \times 10^{-4}$	$1.08848372803124 \times 10^{-2}$
0.53125	0.944424654	$6.01073679791564 \times 10^{-4}$	$9.34223321375005 \times 10^{-3}$
0.59375	0.955684071	$8.11885626383790 \times 10^{-4}$	$7.83318242000008 \times 10^{-3}$
0.65625	0.979119439	$1.02140923121985 \times 10^{-3}$	$6.82823433750002 \times 10^{-3}$
0.71875	1.017672214	$1.22405702226835 \times 10^{-3}$	$7.04804234375001 \times 10^{-3}$
0.78125	1.074350570	$1.41460377351166 \times 10^{-3}$	$9.51275275000008 \times 10^{-3}$
0.84375	1.152180724	$1.58833747131837 \times 10^{-3}$	$1.55906798062502 \times 10^{-2}$
0.90625	1.254160052	$1.74118665842804 \times 10^{-3}$	$2.70451896999999 \times 10^{-2}$
0.96875	1.383212980	$1.86982331763708 \times 10^{-3}$	$4.60788095562501 \times 10^{-2}$

According to the results obtained in Table 1, the Haar Wavelet Method provides better convergence to the exact solution than the Haar Wavelet Collocation Method (see Fig. 1).

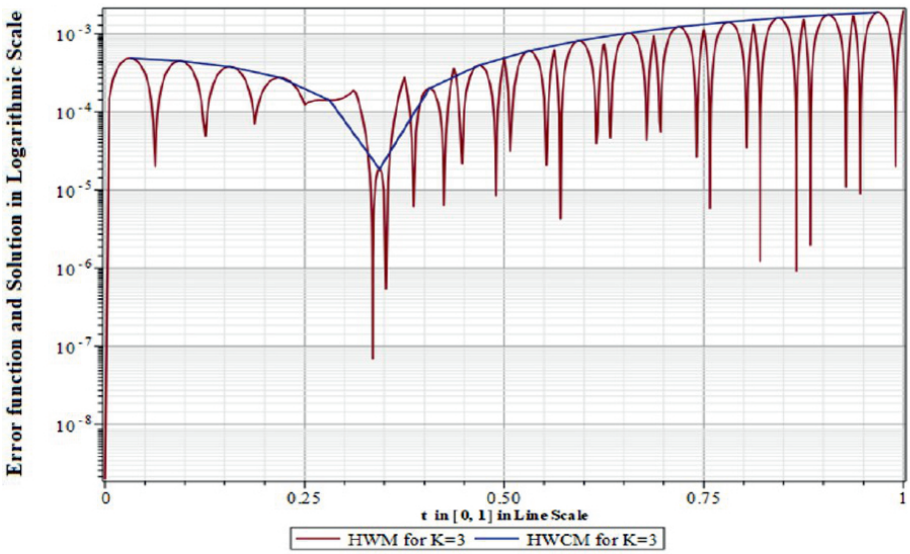

Fig. 1. Comparison of exact solution, Haar wavelet collocation method and Haar wavelet method for $K = 3$ in Example 5.1

Example 5.2. Singularly perturbed two-point boundary value is represented by the following equation:

$$L(z) = -\varepsilon z''(t) + a(t)z'(t) + b(t)z(t) = f(t), \quad z(a) = \alpha, \quad z(b) = \beta,$$

where $t \in [a, b]$ and ε is a small positive parameter. Also $a(t)$, $b(t)$ and $f(t)$ are smooth functions. According to this information, we take the linear boundary value problem

$$-\varepsilon z''(t) + z'(t) + z(t) = 1, \quad z(0) = 0, \quad z(1) = 0$$

with exact solution

$$z(t) = \frac{1}{e^{-B} - e^{A}} \left(e^{-Bt}(e^{A} - 1) - e^{At}(e^{-B} - 1)\right),$$

where $A = \dfrac{1 + \sqrt{1 + 4\varepsilon}}{2\varepsilon}$ and $B = \dfrac{-1 + \sqrt{1 + 4\varepsilon}}{2\varepsilon}$. We now deal with this singular boundary value problem using HWM and HWCM. Firstly, we demonstrate the maximum error for numerical solutions of Haar wavelet method and Haar wavelet collocation method in Table 2.

Besides, the exact solution and numerical approximations for $K = 2, 3, 4$ and $\varepsilon = 0.15$ are shown in Fig. 2.

Table 2

Error analysis of HWM and HWCM

ε	Max Error for HWM		
	$K=2$	$K=3$	$K=4$
0.1	6.7822×10^{-6}	9.335×10^{-7}	6.947208×10^{-4}
0.20	1.1832961×10^{-4}	1.425782×10^{-5}	1.026436×10^{-4}
0.25	1.4688459×10^{-4}	1.782824×10^{-5}	5.49711×10^{-5}
ε	Max Error for HWCM		
	$K=2$	$K=3$	$K=4$
0.1	$5.50251818678332\times10^{-3}$	$3.58750857394036\times10^{-4}$	$1.11261817659303\times10^{-4}$
0.15	$2.88957278491042\times10^{-3}$	$4.39623301760006\times10^{-4}$	$8.73356980457512\times10^{-3}$
0.20	$5.08412287291982\times10^{-3}$	$3.30387375070855\times10^{-3}$	$9.51446394816008\times10^{-4}$
0.25	$1.07875696601853\times10^{-2}$	$1.83906357083652\times10^{-4}$	$1.21884710585993\times10^{-3}$

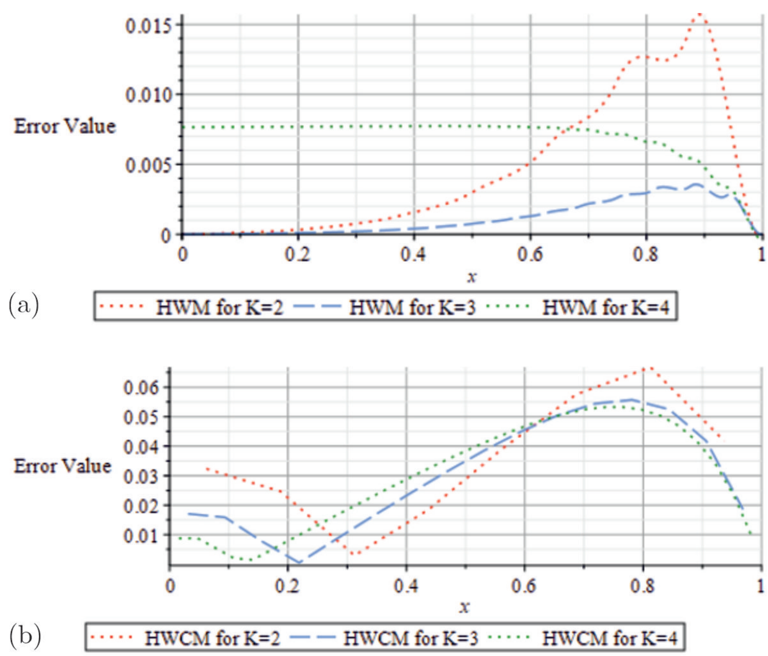

Fig. 2. Exact solutions and approximate solutions of HWM (a) and HWCM (b) at $\varepsilon=0.15$ for $K=2,3,4$ in Example 5.2

6. Conclusion. This study examines the application of Haar wavelets to first and second-order ODE solutions. Its applicability and effectiveness are assessed using two test problems. The obtained results are compared to those obtained by the function and matrix approximation. The Haar wavelet collocation approach provides a more dependable and precise method for solving first and second-order ODEs numerically. By estimating error norms for various problems, the method's effectiveness and accuracy have been shown. It is evident that the function approximation has a very low error rate when comparing the numerical solution of Haar wavelets with two different techniques.

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