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# SOLUTION OF THE BOUNDARY-VALUE PROBLEM OF HEAT CONDUCTION WITH PERIODIC BOUNDARY CONDITIONS

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We investigate the solution of the inverse problem for a linear two-dimensional parabolic equation with periodic boundary and integral overdetermination conditions. Under certain natural regularity and consistency conditions imposed on the input data, we establish the existence, uniqueness of the solution, and its continuous dependence on the data by using the generalized Fourier method. In addition, an iterative algorithm is constructed for the numerical solution of the problem.

## 1. Introduction

The investigations of mathematical models for numerous important applications such as chemical diffusion, applications in heat conduction processes [5, 8], population dynamics, thermoelasticity, medical science, electrochemistry, engineering, wide scope, chemical engineering [9], and control theory require the analyses of two-dimensional parabolic partial differential equations with nonlocal boundary conditions [13, 14, 17].

Inverse problems are defined as problems of finding an unknown property of an object (or a medium) according to the data of observation of the response of this object (medium) to a probing signal. Thus, the theory of inverse problems gives a theoretical basis for remote sensing and nondestructive testing. If, e.g., an acoustic plane wave is scattered by an obstacle and one observes the scattered field far from the obstacle, or in a certain external region, then the inverse problem is to find the shape and material properties of the obstacle. These problems are important for the identification of flying objects (airplanes, missiles, etc.), objects immersed in water (submarines, shoals of fish, etc.), and in many other situations. In geophysics, an acoustic wave is sent from Earth's surface and the scattered field is collected on the surface for various positions of the source of the field at a fixed frequency, or at several frequencies. The inverse problem is to find subsurface inhomogeneities. In technology, the researchers measure the eigenfrequencies of a piece of material, and the inverse problem is to find defects in this material, e.g., hole in the metal. In geophysics, the role of inhomogeneity can be played by oil deposits, caves, mines, etc. In medicine, this may be a tumor or some abnormal objects in the human body. If it is possible to detect inhomogeneities in a medium by analyzing the scattered field on the surface, then it is not necessary to drill holes in the medium. This, in turn, makes it possible to avoid costly and destructive evaluations. The practical advantages of remote sensing make the inverse problems important, as indicated in [20].

There are several methods aimed at the numerical approximation of two-dimensional parabolic inverse problems. In [13], the author considered three different implicit finite-difference schemes for the solution of the two-dimensional parabolic inverse problem with temperature overspecification. These schemes were developed for the identification of the control parameter, which realizes, at any given time, a desired temperature distribution at a given point of the space domain. The discussed numerical methods are based on the second-order backward-time centered-space (BTCS) implicit formula, the second-order Crank–Nicolson implicit finite-difference formula,

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and the fourth-order implicit scheme. These finite difference schemes are unconditionally stable. The implicit formula takes a huge amount of the central processor (CPU) time but its fourth-order accuracy is significant. The results of numerical experiments are presented, and the accuracies and CPU times needed for each of these methods are discussed and compared. The implicit finite-difference schemes use larger amounts of CPU time than the explicit finite-difference techniques but they are stable for every diffusion number.

For the last couple of years, considerable efforts were made to get either approximate analytic solutions or pure numerical solutions of nonlocal boundary-value problems [5, 10–12, 15] and implement finite-difference schemes in order to obtain numerical solutions of one-dimensional nonlocal boundary-value problems [12, 13, 16].

The periodic boundary conditions appear in numerous important applications, such as heat transfer and life sciences [1–4].

In the present paper, we prove the existence, uniqueness, and continuous dependence of the solution on the data and obtain the numerical solution of the two-dimensional diffusion problem with periodic boundary conditions. We apply the Fourier method and the finite-difference method for the two-dimensional inverse parabolic equation [1–3].

The paper is organized as follows. In Section 2, we prove the existence and uniqueness of the solution of the problem by using the Fourier method. In Section 3, we demonstrate the stability of the solution. In Section 4, we present a numerical procedure used for the solution of the problem.

Let $T > 0$ be a fixed number. We denote

$$\Omega := \{(x, y, t)\colon 0 < x < \pi,\ 0 < y < \pi,\ 0 < t < T\}.$$

Consider a problem of finding a pair of functions $\{r(t),\ u(x, y, t)\}$ satisfying the following equations:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + r(t) f(x, y, t), \quad (x, y, t) \in \Omega, \tag{1}$$

$$\begin{aligned} u(0, y, t) &= u(\pi, y, t), \qquad y \in [0, \pi], \quad t \in [0, T], \\ u(x, 0, t) &= u(x, \pi, t), \qquad x \in [0, \pi], \quad t \in [0, T], \end{aligned} \tag{2}$$

$$\begin{aligned} u_x(0, y, t) &= u_x(\pi, y, t), \qquad y \in [0, \pi], \quad t \in [0, T], \\ u_y(x, 0, t) &= u_y(x, \pi, t), \qquad x \in [0, \pi], \quad t \in [0, T], \end{aligned} \tag{3}$$

$$u(x, y, 0) = \varphi(x, y), \qquad x \in [0, \pi], \quad y \in [0, \pi], \tag{4}$$

$$E(t) = \int_0^\pi \int_0^\pi xyu(x, y, t)\, dx\, dy, \quad t \in [0, T], \tag{5}$$

for a two-dimensional parabolic equation with periodic boundary conditions. The functions $\varphi(x, y)$ and $f(x, y, t)$ are given on $[0, \pi] \times [0, \pi]$ and $\overline{\Omega}$, respectively. In the process of heat propagation in a thin rod, the law of variation $E(t)$ of the total amount of heat in the rod is given in [18]. This integral condition in parabolic problems is also called the heat-moments condition. The heat moments are analyzed in [19].

Condition (4) is an initial condition. Conditions (2) and (3) are periodic Dirichlet and Neumann conditions, respectively.

Problem (1)–(5) is called an inverse problem. The pair

$$\{r(t), u(x,y,t)\}$$

from the class $C[0,T] \times \left(C^{2,2,1}(\Omega) \cap C^{1,1,0}(\overline{\Omega})\right)$ for which conditions (1)–(5) are satisfied is called the classical solution of the inverse problem (1)–(5).

The inverse problem of finding the heat source in a parabolic equation was investigated in numerous works for the cases where the unknown heat source is space-dependent [6, 7] and time-dependent [5].

**Notation:**

$\varphi(x,y)$ is the initial temprature,

$r(t)$ is an unknown coefficient,

$E(t)$ is energy,

$u(x,y,t)$ is the temperature distribution,

$f(x,y,t)$ is a source function,

$u_0(t),\ u_{cmn}(t),\ u_{csmn}(t),\ u_{scmn}(t),\ u_{smn}(t)$ are the Fourier coefficients,

$M$ is an arbitrary constant,

$M_1,\ M_2,\ M_3,\ M_4,\ M_5,\ M_6$ are dimensionless constants,

$F(t)$ is a continuous function, $K(t,\tau)$ is a kernel function,

$\Omega := \{(x,y,t)\colon\ 0 < x < \pi,\ 0 < y < \pi,\ 0 < t < T\}$ is the domain of $x,\ y,\ t$.

## 2. Existence and Uniqueness of the Solution of Inverse Problem

We seek the solution of (1)–(5) in the following form:

$$\begin{aligned} u(x,y,t) = \frac{u_0(t)}{4} &+ \sum_{m,n=1}^{\infty} u_{cmn}(t)\cos 2mx \cos 2ny \\ &+ \sum_{m,n=1}^{\infty} u_{csmn}(t)\cos 2mx \sin 2ny \\ &+ \sum_{m,n=1}^{\infty} u_{scmn}(t)\sin 2mx \cos 2ny \\ &+ \sum_{m,n=1}^{\infty} u_{smn}(t)\sin 2mx \sin 2ny. \end{aligned}$$

By applying the standard procedure of the Fourier method, we obtain the Fourier coefficients in the form

$$u_0(t) = \varphi_0 + \frac{4}{\pi^2}\int_0^t\int_0^\pi\int_0^\pi r(\tau) f(\xi,\eta,\tau)\, d\xi\, d\eta\, d\tau,$$

$$u_{cmn}(t) = \varphi_{cmn} e^{-((2m)^2+(2n)^2)t}$$
$$+ \frac{4}{\pi^2}\int_0^t\int_0^\pi\int_0^\pi r(\tau) e^{-((2m)^2+(2n)^2)(t-\tau)} f(\xi,\eta,\tau) \cos 2m\xi \cos 2n\eta\, d\xi\, d\eta\, d\tau,$$

$$u_{csmn}(t) = \varphi_{csmn} e^{-((2m)^2+(2n)^2)t}$$
$$+ \frac{4}{\pi^2}\int_0^t\int_0^\pi\int_0^\pi r(\tau) e^{-((2m)^2+(2n)^2)(t-\tau)} f(\xi,\eta,\tau) \cos 2m\xi \sin 2n\eta\, d\xi\, d\eta\, d\tau,$$

$$u_{scmn}(t) = \varphi_{scmn} e^{-((2m)^2+(2n)^2)t}$$
$$+ \frac{4}{\pi^2}\int_0^t\int_0^\pi\int_0^\pi r(\tau) e^{-((2m)^2+(2n)^2)(t-\tau)} f(\xi,\eta,\tau) \sin 2m\xi \cos 2n\eta\, d\xi\, d\eta\, d\tau,$$

$$u_{smn}(t) = \varphi_{smn} e^{-((2m)^2+(2n)^2)t}$$
$$+ \frac{4}{\pi^2}\int_0^t\int_0^\pi\int_0^\pi r(\tau) e^{-((2m)^2+(2n)^2)(t-\tau)} f(\xi,\eta,\tau) \sin 2m\xi \sin 2n\eta\, d\xi\, d\eta\, d\tau,$$

where

$$\varphi_0 = \frac{4}{\pi^2}\int_0^\pi\int_0^\pi \varphi(x,y)\, dx\, dy,$$

$$\varphi_{cmn} = \frac{4}{\pi^2}\int_0^\pi\int_0^\pi \varphi(x,y) \cos 2mx \cos 2ny\, dx\, dy,$$

$$\varphi_{csmn} = \frac{4}{\pi^2}\int_0^\pi\int_0^\pi \varphi(x,y) \cos 2mx \sin 2ny\, dx\, dy,$$

$$\varphi_{scmn} = \frac{4}{\pi^2}\int_0^\pi\int_0^\pi \varphi(x,y) \sin 2mx \cos 2ny\, dx\, dy,$$

$$\varphi_{smn} = \frac{4}{\pi^2}\int_0^\pi\int_0^\pi \varphi(x,y) \sin 2mx \sin 2ny\, dx\, dy.$$

We obtain the following solution of the problem (1)–(4) for any $r(t) \in C[0,T]$ :

$$
\begin{aligned}
u(x,y,t) = \frac{1}{4}\left(\varphi_0 + \frac{4}{\pi^2}\int_0^t r(\tau) f_0(\tau) d\xi d\eta d\tau\right) \\
+ \sum_{m,n=1}^{\infty}\Bigg(\varphi_{cmn} e^{-((2m)^2+(2n)^2)t} \\
+ \frac{4}{\pi^2}\int_0^t r(\tau) e^{-((2m)^2+(2n)^2)(t-\tau)} f_{cmn}(\tau) d\tau\Bigg)\cos 2mx \cos 2ny \\
+ \sum_{m,n=1}^{\infty}\Bigg(\varphi_{csmn} e^{-((2m)^2+(2n)^2)t} \\
+ \frac{4}{\pi^2}\int_0^t r(\tau) e^{-((2m)^2+(2n)^2)(t-\tau)} f_{csmn}(\tau)\, d\tau\Bigg)\cos 2mx \sin 2ny \\
+ \sum_{m,n=1}^{\infty}\Bigg(\varphi_{scmn} e^{-((2m)^2+(2n)^2)t} \\
+ \frac{4}{\pi^2}\int_0^t r(\tau) e^{-((2m)^2+(2n)^2)(t-\tau)} f_{scmn}(\tau)\, d\tau\Bigg)\sin 2mx \cos 2ny \\
+ \sum_{m,n=1}^{\infty}\Bigg(\varphi_{smn} e^{-((2m)^2+(2n)^2)t} \\
+ \frac{4}{\pi^2}\int_0^t r(\tau) e^{-((2m)^2+(2n)^2)(t-\tau)} f_{smn}(\tau)\, d\tau\Bigg)\sin 2mx \sin 2ny,
\end{aligned}
\tag{6}
$$

where

$$
f_0(t) = \frac{4}{\pi^2}\int_0^{\pi}\int_0^{\pi} f(x,y,t)\, dx\, dy,
$$

$$
f_{cmn}(t) = \frac{4}{\pi^2}\int_0^{\pi}\int_0^{\pi} f(x,y,t)\cos 2mx \cos 2ny\, dx\, dy,
$$

$$f_{csmn}(t) = \frac{4}{\pi^2} \int_0^\pi \int_0^\pi f(x,y,t) \cos 2mx \sin 2ny \, dx \, dy,$$

$$f_{scmn}(t) = \frac{4}{\pi^2} \int_0^\pi \int_0^\pi f(x,y,t) \sin 2mx \cos 2ny \, dx \, dy,$$

$$f_{smn}(t) = \frac{4}{\pi^2} \int_0^\pi \int_0^\pi f(x,y,t) \sin 2mx \sin 2ny \, dx \, dy.$$

**Theorem 1.** *Suppose that the following conditions are satisfied:*

*($A_1$)* $E(t) \in C^1[0,T]$,

*($A_2$)* $\varphi(x,y) \in C^{2,2}\big([0,\pi]\times[0,\pi]\big)$, $\varphi(0,y) = \varphi(\pi,y)$, $\varphi_x(0,y) = \varphi_x(\pi,y)$, $\varphi(x,0) = \varphi(x,\pi)$, $\varphi_y(x,0) = \varphi_y(x,\pi)$, *and*

$$\int_0^\pi \int_0^\pi xy\varphi(x,y) \, dx \, dy = E(0),$$

*($A_3$)* $f(x,y,t) \in C^{2,2,0}(\overline{\Omega})$, $f(0,y,t) = f(\pi,y,t)$, $f_x(0,y,t) = f_x(\pi,y,t)$, $f(x,0,t) = f(x,\pi,t)$, $f_y(x,0,t) = f_y(x,\pi,t)$, *and*

$$\int_0^\pi \int_0^\pi xyf(x,y,t) \, dx \, dy \neq 0.$$

*Then system (1)–(5) possesses a unique solution.*

***Proof.*** The assumptions

$$\varphi(0,y) = \varphi(\pi,y), \qquad \varphi(x,0) = \varphi(x,\pi), \qquad f(0,y,t) = f(\pi,y,t), \qquad f(x,0,t) = f(x,\pi,t)$$

are the consistency conditions necessary for the solution $u(x,y,t)$ to belong to $C^{2,2,1}(\Omega) \cap C^{1,1,0}\big(\overline{\Omega}\big)$. Further, under the smoothness assumptions

$$\varphi(x,y) \in C^{2,2}\big([0,\pi]\times[0,\pi]\big) \quad \text{and} \quad f(x,y,t) \in C^{2,2}\big([0,\pi]\times[0,\pi]\big) \quad \forall t \in [0,T],$$

series (6) and its $x,y$-partial derivative converge uniformly in $\overline{\Omega}$ because their majorizing sums are absolutely convergent. Therefore, their sums $u(x,y,t), u_x(x,y,t)$, and $u_y(x,y,t)$ are continuous in $\overline{\Omega}$. In addition, the $t$-partial derivative and the $xx$- and $yy$-second-order partial derivative series are uniformly convergent for $t > 0$. Thus,

$$u(x,y,t) \in C^{2,2,1}(\Omega) \cap C^{1,1,0}(\overline{\Omega})$$

and satisfies conditions (1)–(5). In addition, $u_t(x,y,t)$ is continuous in $\overline{\Omega}$ because the majorizing sum of the $t$-partial derivative series is absolutely convergent under the conditions $\varphi_x(0,y) = \varphi_x(\pi,y)$, $\varphi_y(x,0) = \varphi_y(x,\pi)$, $f_x(0,y,t) = f_x(\pi,y,t)$, and $f_y(x,0,t) = f_y(x,\pi,t)$ in $\overline{\Omega}$.

We differentiate equation (5) under the condition ($A_1$) to obtain

$$E'(t) = \int_0^\pi \int_0^\pi xyu_t(x,y,t)\,dx\,dy. \tag{7}$$

Further, under the consistency assumption

$$\int_0^\pi \int_0^\pi xy\varphi(x,y)\,dx\,dy = E(0),$$

relations (6) and (7) yield the following Volterra integral equation of the second kind:

$$r(t) = F(t) + \int_0^t K(t,\tau)r(\tau)\,d\tau, \quad t \in [0,T],$$

where

$$F(t) = \frac{E'(t) + \frac{\pi^2}{4}\sum_{m,n=1}^{\infty} \frac{(2m)^2+(2n)^2}{mn}\varphi_{smn}e^{-\left((2m)^2+(2n)^2\right)t}}{f_0(t) + \frac{\pi^2}{4}\sum_{m,n=1}^{\infty} \frac{(2m)^2+(2n)^2}{mn}f_{smn}(t)}, \tag{8}$$

$$K(t,\tau) = \frac{\frac{\pi^2}{4}\sum_{m,n=1}^{\infty} \frac{(2m)^2+(2n)^2}{mn}f_{smn}(\tau)e^{-\left((2m)^2+(2n)^2\right)(t-\tau)}}{f_0(t) + \frac{\pi^2}{4}\sum_{m,n=1}^{\infty} \frac{(2m)^2+(2n)^2}{mn}f_{smn}(t)}. \tag{9}$$

Under the assumption ($A_1$)–($A_3$), the function $F(t)$ and the kernel function $K(t,\tau)$ are functions continuous in $[0,T]$ and $[0,T]\times[0,T]$, respectively. Thus, we obtain a unique function $r(t)$ continuous on $[0,T]$. Together with the solution of problem (1)–(4) given by the Fourier series $u(x,y,t)$, this solution forms the unique solution of the inverse problem (1)–(5).

Theorem 1 is proved.

## 3. Continuous Dependence of $(r,u)$ upon the Data

We have the following result on the continuous dependence of solution of the inverse problem (1)–(5) on the data:

**Theorem 2.** *If* $\Phi = \{\varphi, E, f\}$ *satisfies the assumptions ($A_1$)–($A_3$) of Theorem 1, then the solution* $(r,u)$ *of problem (1)–(5) continuously depends on the data* $\varphi$, $E$, *and* $f$.

***Proof.*** Let $\Phi = \{\varphi, E, f\}$ and $\overline{\Phi} = \{\overline{\varphi}, \overline{E}, \overline{f}\}$ be two sets of data satisfying the assumptions (A1)–(A3). Suppose that there exist positive constants $M_i,\ i = 1,2,3,4,5,$ such that

$$\|f\|_{C^{2,2,0}(\Omega)} \le M, \qquad \|\overline{f}\|_{C^{2,2,0}(\Omega)} \le M,$$

$$\|\varphi\|_{C^{2,2}([0,\pi]\times[0,\pi])} \le M_1, \qquad \|\overline{\varphi}\|_{C^{2,2}([0,\pi]\times[0,\pi])} \le M_1,$$

$$\|E\|_{C^1[0,T]} \le M_2, \qquad \|\overline{E}\|_{C^1[0,T]} \le M_2,$$

$$\|F\|_{C[0,T]} \le M_3, \qquad \|\overline{F}\|_{C[0,T]} \le M_3,$$

$$\|K\|_{C([0,T]\times[0,T])} \le M_4,$$

$$0 < M_5 \le \min_{(x,y,t)\in\overline{\Omega}} |f(x,y,t)|, \qquad 0 < M_5 \le \min_{(x,y,t)\in\overline{\Omega}} \left|\overline{f}(x,y,t)\right|.$$

We denote

$$\|\Phi\| = \Big(\|E\|_{C^1[0,T]} + \|\varphi\|_{C^{2,2}([0,\pi]\times[0,\pi])} + \|f\|_{C^{2,2,0}(\overline{\Omega})}\Big).$$

Let $(r,u)$ and $(\overline{r},\overline{u})$ be the solutions of inverse problems (1)–(5) corresponding to the data

$$\Phi = \{\varphi, E, f\} \quad \text{and} \quad \overline{\Phi} = \{\overline{\varphi}, \overline{E}, \overline{f}\},$$

respectively, where

$$\begin{aligned}\|\varphi-\overline{\varphi}\|_{C^{2,2}([0,\pi]\times[0,\pi])} \le{}& \frac{\|\varphi_0-\overline{\varphi_0}\|_{C^{2,2}([0,\pi]\times[0,\pi])}}{4} \\ &+ \frac{1}{6}\sum_{m,n=1}^{\infty}\left\|(\varphi_{xy})_{cmn} - \overline{(\varphi_{xy})_{cmn}}\right\|_{C^{2,2}([0,\pi]\times[0,\pi])} \\ &+ \frac{1}{6}\left\|(\varphi_{xy})_{csmn} - \overline{(\varphi_{xy})_{csmn}}\right\|_{C^{2,2}([0,\pi]\times[0,\pi])} \\ &+ \frac{1}{6}\left\|(\varphi_{xy})_{scmn} - \overline{(\varphi_{xy})_{scmn}}\right\|_{C^{2,2}([0,\pi]\times[0,\pi])} \\ &+ \frac{1}{6}\left\|(\varphi_{xy})_{smn} - \overline{(\varphi_{xy})_{smn}}\right\|_{C^{2,2}([0,\pi]\times[0,\pi])},\end{aligned}$$

$$(\varphi_{xy})_{cmn} = \frac{4}{\pi^2 mn}\int_0^{\pi}\int_0^{\pi}\varphi_{xy}(x,y)\sin 2mx \sin 2ny\,dx\,dy,$$

$$(\varphi_{xy})_{csmn} = \frac{4}{\pi^2 mn}\int_0^{\pi}\int_0^{\pi}\varphi_{xy}(x,y)\sin 2mx \cos 2ny\,dx\,dy,$$

$$(\varphi_{xy})_{scmn} = \frac{4}{\pi^2 mn}\int_0^{\pi}\int_0^{\pi}\varphi_{xy}(x,y)\cos 2mx \sin 2ny\,dx\,dy,$$

$$(\varphi_{xy})_{smn} = \frac{4}{\pi^2 mn} \int_0^{\pi}\int_0^{\pi} \varphi_{xy}(x,y)\cos 2mx \cos 2ny \, dx \, dy.$$

It follows from (10) that

$$F(t) - \overline{F(t)} = \frac{E'(t) + \frac{\pi^2}{4}\sum_{m,n=1}^{\infty} \frac{(2m)^2+(2n)^2}{mn}\varphi_{smn} e^{-((2m)^2+(2n)^2)t}}{f_0(t) + \frac{\pi^2}{4}\sum_{m,n=1}^{\infty} \frac{(2m)^2+(2n)^2}{mn} f_{smn}(t)}$$

$$- \frac{\overline{E}'(t) + \frac{\pi^2}{4}\sum_{m,n=1}^{\infty} \frac{(2m)^2+(2n)^2}{mn}\overline{\varphi}_{smn} e^{-((2m)^2+(2n)^2)t}}{\overline{f}_0(t) + \frac{\pi^2}{4}\sum_{m,n=1}^{\infty} \frac{(2m)^2+(2n)^2}{mn} \overline{f}_{smn}(t)}.$$

Applying Hölder's inequality and taking the maximum values of both sides of the last inequality, we obtain

$$\left\|F - \overline{F}\right\| \le \frac{2M}{M_5^2}\left\|E'(t) - \overline{E'(t)}\right\| + \frac{2M\pi^3}{\sqrt{6}M_5^2}\left\|\varphi - \overline{\varphi}\right\| + M_0\left\|f - \overline{f}\right\|$$

and, similarly,

$$\left\|K - \overline{K}\right\| \le M_6\left\|f - \overline{f}\right\|,$$

$$\left\|r - \overline{r}\right\| \le \left\|F - \overline{F}\right\| + M_4 T\left\|r - \overline{r}\right\| + \frac{MT}{1 - TM_4}\left\|K - \overline{K}\right\|,$$

$$\left\|r - \overline{r}\right\| \le \frac{2M}{M_5^2(1 - TM_4)}\left\|E'(t) - \overline{E'(t)}\right\| + \frac{M_0}{1 - TM_4}\left\|f - \overline{f}\right\|$$

$$+ \frac{TM}{(1 - TM_4)^2}\left\|f - \overline{f}\right\| + \frac{2M\pi^3}{\sqrt{6}M_5^2(1 - TM_4)}\left\|\varphi - \overline{\varphi}\right\|,$$

$$\left\|u - \overline{u}\right\| \le M_7\left\|E'(t) - \overline{E'(t)}\right\| + M_8\left\|\varphi - \overline{\varphi}\right\| + M_9\left\|f - \overline{f}\right\|,$$

$$\left\|u - \overline{u}\right\| \le M_{10}\left\|\Phi - \overline{\Phi}\right\|,$$

where

$$M_0 = \max\left\{\frac{\pi^4 M_2}{6M_5^2}, \frac{\pi^3 M_2}{\sqrt{6}M_5^2}\right\}, \qquad M_6 = \max\left\{\frac{\pi^3 2M}{\sqrt{6}M_5^2}, \qquad \frac{\pi^3}{\sqrt{6}M_5^2}\right\},$$

$$M_7 = \frac{2MT}{3M_5^2(1 - TM_4)}, \qquad M_8 = \max\left\{1, \frac{4M^2\pi^3 T}{3\sqrt{6}M_5^2(1 - TM_4)}\right\},$$

$$M_9 = \max\left\{\frac{2MT}{3M_5^2(1-TM_4)}, \frac{2M^2M_6T^2}{3(1-TM_4)}\right\}, \qquad M_{10} = \max\{1, M_7, M_8, M_9\}.$$

If $\Phi \to \overline{\Phi}$ then $r \to \overline{r}$ and $u \to \overline{u}$.
Theorem 2 is proved.

## 4. Numerical Method for the Solution of Problem (1)–(4)

In this section, we use the implicit finite-difference approximation of the discretized problem (1)–(5):

$$\begin{gathered}
\frac{1}{\tau}\left(u_{i,j}^{k+1} - u_{i,j}^{k}\right) = \frac{1}{h^2}\left(u_{i-1,j}^{k+1} - 2u_{i,j}^{k+1} + u_{i+1,j}^{k+1}\right) \\
+ \frac{1}{h^2}\left(u_{i,j-1}^{k+1} - 2u_{i,j}^{k+1} + u_{i,j+1}^{k+1}\right) + r^{k+1} f_{i,j}^{k+1}, \\
u_{i,j}^{0} = \phi_i, \\
u_{0,j}^{k} = u_{M+1,j}^{k}, \qquad u_{M+1,j}^{k} = \frac{u_{1,j}^{k} - u_{M,j}^{k}}{2}, \\
u_{i,0}^{k} = u_{i,M+1}^{k}, \qquad u_{i,M+1}^{k} = \frac{u_{i,1}^{k} - u_{i,M}^{k}}{2},
\end{gathered} \tag{10}$$

where the computational domain $[0,\pi] \times [0,\pi] \times [0,T]$ is discretized as follows:

$$x_i = ih, \quad i = 0, 1, \ldots, M, \qquad y_j = jh, \quad j = 0, 1, \ldots, M, \qquad \text{and} \qquad t_k = k\tau, \quad k = 0, 1, \ldots, N,$$

where $h = \pi/M$ and $\tau = T/N$ are the space and time steps, respectively, $M$ and $N$ are two positive integers,

$$u_{i,j}^{k} = u(x_i, y_j, t_k), \qquad f_{i,j}^{k} = f(x_i, y_j, t_k), \qquad \text{and} \qquad r^k = r(t_k).$$

We now integrate equation (1) with respect to $x$ and $y$ from $0$ to $\pi$ and apply relations (2), (3) and (5). This yields

$$r(t) = \frac{E'(t) - \displaystyle\int_0^\pi y u_x(\pi, y, t)\, dy - \int_0^\pi x u_y(x, \pi, t)\, dx}{\displaystyle\int_0^\pi \int_0^\pi xyf(x, y, t)\, dx\, dy}. \tag{11}$$

The finite-difference approximation of relation (11) is

$$r^{k+1} = \frac{(E^{k+2} - E^k)/\tau - \left(\displaystyle\int_0^\pi y u_x(\pi, y, t)\, dy\right)^k - \left(\displaystyle\int_0^\pi x u_y(x, \pi, t)\, dx\right)^k}{\left(\displaystyle\int_0^\pi \int_0^\pi xyf(x, y, t)\, dx\, dy\right)^k},$$

where $E^k = E(t_k),\ k = 0, 1, \ldots, N$. Note that the integrals are numerically computed by using Simpson's rule of integration and the first derivatives are taken with the help of the central difference scheme.

Let $r^{k(s)},\ u_{i,j}^{k(s)}$ be the values of $r^k$ and $u_{i,j}^k$ at the $s$th iteration step, respectively. At each $(s+1)$th iteration step, the value of $r^{k+1(s+1)}$ is given by the formula

$$r^{k+1(s+1)} = \frac{(E^{k+2} - E^k)/\tau - \left(\displaystyle\int_0^\pi y u_x(\pi, y, t)\, dy\right)^{k(s)} - \left(\displaystyle\int_0^\pi x u_y(x, \pi, t)\, dx\right)^{k(s)}}{\left(\displaystyle\int_0^\pi \int_0^\pi xy f(x, y, t)\, dx\, dy\right)^k}.$$

For the iteration of (10), we get

$$\begin{gathered}
\frac{1}{\tau}\left(u_{i,j}^{k+1(s+1)} - u_{i,j}^{k+1(s)}\right) = \frac{1}{h^2}\left(u_{i-1,j}^{k+1(s+1)} - 2u_{i,j}^{k+1(s+1)} + u_{i+1,j}^{k+1(s+1)}\right) \\
+ \frac{1}{h^2}\left(u_{i,j-1}^{k+1(s+1)} - 2u_{i,j}^{k+1(s+1)} + u_{i,j+1}^{k+1(s+1)}\right) + r^{k+1(s+1)} f_{i,j}^{k+1}, \\
u_{i,j}^0 = \phi_i, \\
u_{0,j}^{k+1(s)} = u_{M+1,j}^{k+1(s)}, \qquad u_{M+1,j}^{k+1(s)} = \frac{u_{1,j}^{k+1(s)} - u_{M,j}^{k+1(s)}}{2}, \\
u_{i,0}^{k+1(s)} = u_{i,M+1}^{k+1(s)}, \qquad u_{i,M+1}^{k+1(s)} = \frac{u_{i,1}^{k+1(s)} - u_{i,M}^{k+1(s)}}{2}.
\end{gathered} \tag{12}$$

The system of equations (12) is solved and $u_{i,j}^{k+1(s+1)}$ is determined. If the difference between the values obtained for two iterations reaches the prescribed tolerance, the iterative process is terminated.

In order to illustrate the behavior of our numerical method, we consider the following example:

***Example 1.*** In this example, we investigate the process of finding of the exact solution

$$\{r(t),\ u(x, y, t)\} = \{2\exp(t^2),\ (2 + \cos 2x + \cos 2y)\exp(t^2)\}$$

for the given functions

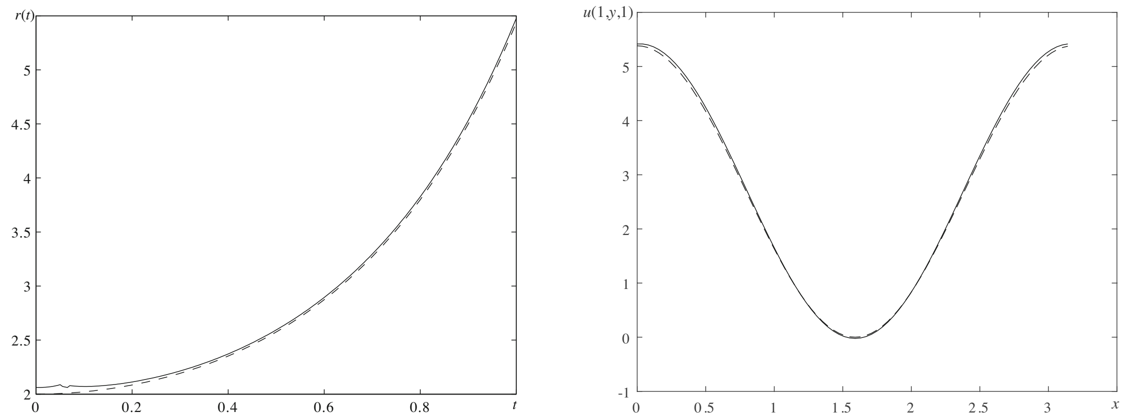
$$\varphi(x, y) = (2 + \cos 2x + \cos 2y), \qquad E(t) = \frac{\pi^4}{2}\exp(t^2),$$

$$F(x, y, t) = 2t + (t + 2)(\cos 2x + \cos 2y).$$

The step sizes are $h = 0.0393$ and $\tau = 0.005$.

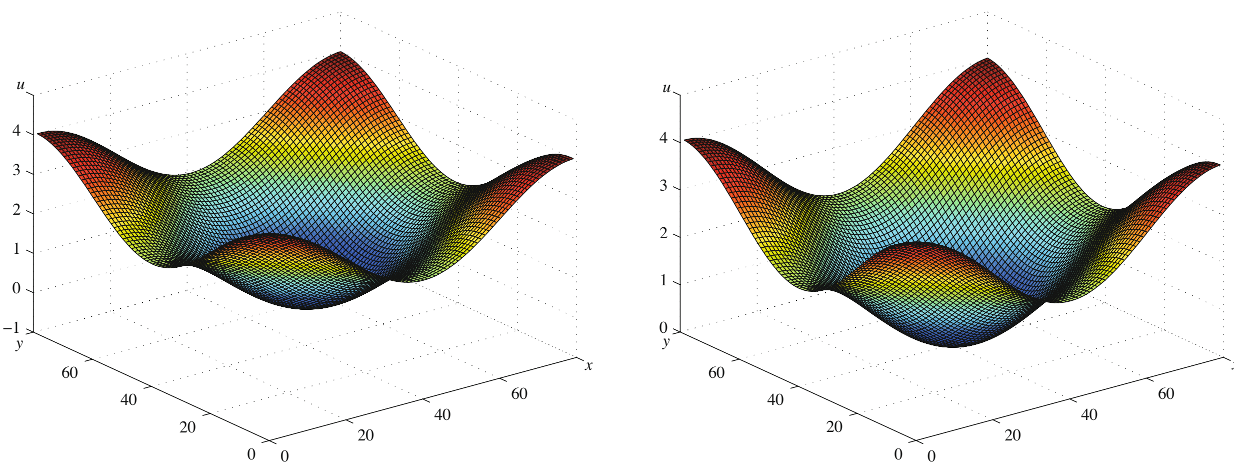
Note that the convergence criterion for $r(t)$ is as follows:

$$\left|r^{k+1(s+1)} - r^{k+1(s)}\right| \le h/200.$$




**(a)** **(b)**

**Fig. 1.** The exact and approximate solutions for $r(t)$ (a) and $u(1, y, 1)$ (b). The exact solution is shown by the dashed line.




**(a)** **(b)**

**Fig. 2.** The approximate (a) and numerical (b) solutions for $u(x, y, 1/10)$.

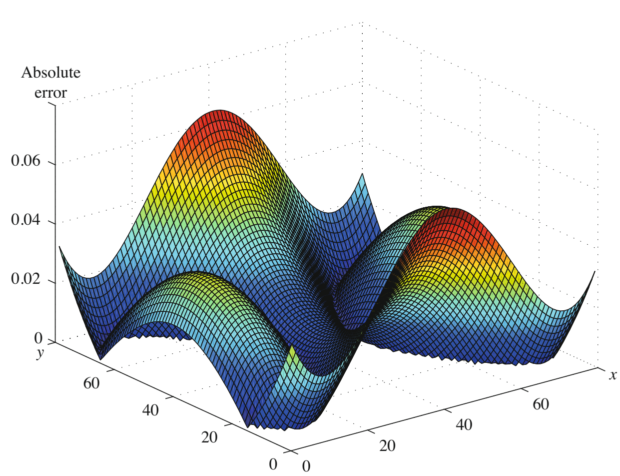



**Fig. 3.** The absolute error of $u(x, y, 1/10)$.

**Table 1**
**Some Values of $r(t)$**

| Exact | Approximate | Error | Relative error |
|---|---|---|---|
| 2 | 2.0614 | 0.0614 | 0.0307 |
| 2.0201 | 2.0717 | 0.0516 | 0.0255 |
| 2.0816 | 2.1098 | 0.0282 | 0.0135 |
| 2.1883 | 2.2099 | 0.0216 | 0.0099 |
| 2.3470 | 2.3664 | 0.0194 | 0.0083 |
| 2.5681 | 2.5874 | 0.0193 | 0.0075 |
| 2.8667 | 2.8873 | 0.0206 | 0.0072 |
| 3.2646 | 3.2878 | 0.0232 | 0.0071 |
| 3.7930 | 3.8201 | 0.0272 | 0.0072 |
| 4.4958 | 4.5287 | 0.0410 | 0.0075 |

The results of comparison of the exact solution with the numerical finite-difference solutions are shown in Figs. 1–3 and Table 1 with $T = 1$.

## REFERENCES

1. I. Baglan, "Determination of a coefficient in a quasilinear parabolic equation with periodic boundary condition," *Inverse Probl. Sci. Eng.* (2014).
2. F. Kanca, "The inverse coefficient problem of the heat equation with periodic boundary and integral overdetermination conditions," *J. Inequal. Appl.* (2013).
3. F. Kanca and I. Baglan, "An inverse coefficient problem for a quasilinear parabolic equation with nonlocal boundary conditions," *Bound. Value Probl.*, **213** (2013).
4. G. W. Hill, "On the part of the motion of the lunar perigee which is a function of the mean motions of the sun and moon," *Acta Math.*, **8**, 1–36 (1886).
5. J. R. Cannon, "The solution of the heat conduction subject to the specification of energy," *Quart. Appl. Math.*, **21**, 155–160 (1983).
6. J. R. Cannon, "Determination of an unknown heat source from overspecified boundary data," *SIAM J. Numer. Anal.*, **5**, No. 2, 275–286 (1968).
7. F. Dou, C.-L. Fu, and F. Yang, "Identifying an unknown source term in a heat equation," *Inverse Probl. Sci. Eng.*, **17**, No. 7, 901–913 (2009).
8. P. R. Sharma and G. Methi, "Solution of two dimensional parabolic equation subject to non-local conditions using homotopy perturbation method," *J. Comput. Sci. Appl.*, **1**, 12–16 (2012).
9. Y. S. Choi and K. Y. Chan, "A parabolic equation with a nonlocal boundary condition arising from electrochemistry," *Nonlin. Anal.*, **18**, 317–331 (1992).
10. E. Ozbilge and A. Demir, "Inverse problem for a time-fractional parabolic equation," *J. Inequal. Appl.* (2015).
11. E. Ozbilge and A. Demir, "Identification of unknown coefficient in time fractional parabolic equation with mixed boundary conditions via semigroup approach," *Dynam. Syst. Appl.*, **24**, No. 3, 341–348 (2015).
12. M. Dehghan, "Finite difference schemes for two-dimensional parabolic inverse problem with temperature overspecification," *Int. J. Comput. Math.*, **75**, No. 3, 339–349 (2007).
13. M. Dehghan, "Implicit locally one-dimensional methods for two-dimensional diffusion with a nonlocal boundary condition," *Math. Comput. Simulat.*, **49**, 331–349 (1999).
14. S. Wang and Y. Lin, "A finite difference solution to an inverse problem determining a controlling function in a parabolic partial differential equation," *Inverse Probl.*, **5**, 631–640 (1989).

15. A. Mohebbi and M. Abbasi, "A fourth-order compact difference scheme for the parabolic inverse problem with an overspecification at a point," *Inverse Probl. Sci. Eng.*, **23**, No. 3, 457–478 (2014).
16. R. Zolfaghari, "Parameter determination in a parabolic inverse problem in general dimensions," *Comput. Methods Differ. Equat.*, **1**, No. 1, 55–70 (2013).
17. A. Mohebbi, "A numerical algorithm for determination of a control parameter in two-dimensional parabolic inverse problems," *Acta Math. Appl. Sinica*, **31**, No. 1, 213–224 (2015).
18. N. I. Ionkin, "Solution of a boundary-value problem in heat conduction with a nonclassical boundary condition," *Different. Equat.*, **3**, 204–211 (1977).
19. M. Ivanchov and N. Pabyrivska, "Simultaneous determination of two coefficients of a parabolic equation in the case of nonlocal and integral conditions," *Ukr. Math. Zh.*, **53**, No. 5, 589–596 (2001); ***English translation:*** *Ukr. Math. J.*, **53**, No. 5, 674–684 (2001).
20. G. Ramm, *Mathematical and Analytical Techniques with Application to Engineering*, Springer, New York (2005).